Proof Reuse in Logical Frameworks

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- . . . which of these are subsystems of one another?
- . . . what translations exist from one to another?
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I am interested in proof assistants as tools for experimenting with systems of logic.
How well do proof assistants offer machine support for this work?

Not Very Well

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The usual plan of attack:

- Choose one system of logic.
- Implement a proof checker for that one system.
- Build up a big library of formalised results in that system.

We have to start from scratch with each new system.

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The usual plan of attack:

- Choose one system of logic.
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We have to start from scratch with each new system.
There are also logical frameworks (Isabelle, TWELF, Plastic, . . .)

- They can implement more than one system of logic.
- But there is no easy way to use results from one system when working in another.

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There exist *sound translations* between logical systems.

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A *sound translation* from S to T is a mapping

$$\Phi : \text{propositions of } S \to \text{ propositions of } T$$

such that

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I want to:

- declare S and T in some logical framework;
- prove α in S
- and immediately have $\Phi(\alpha)$ available when working in T.

Arithmetic in LF

To declare *Heyting arithmetic* in LF, we give ourselves:

- a kind **Term**, and constants

$$0 : \mathbf{Term}, \qquad S : \mathbf{Term} \to \mathbf{Term},$$
$$+ : \mathbf{Term} \to \mathbf{Term} \to \mathbf{Term}, \qquad \times : \mathbf{Term} \to \mathbf{Term} \to \mathbf{Term} \ ;$$

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$$=_I : \mathbf{Term} \to \mathbf{Term} \to \mathbf{Prop}_I, \qquad \vee_I : \mathbf{Prop}_I \to \mathbf{Prop}_I \to \mathbf{Prop}_I,$$
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- constants for the rules of deduction:

$$\frac{\begin{array}{ccc} [P] & [Q] & \\ \vdots & \vdots & \\ R & R & P \vee Q \end{array}}{R}$$

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$$\begin{aligned} \vee E \ : \ & (P, Q, R : \mathbf{Prop}_I) \\ & (\mathrm{Prf}\,(P) \to \mathrm{Prf}\,(R)) \to \\ & (\mathrm{Prf}\,(Q) \to \mathrm{Prf}\,(R)) \to \\ & \mathrm{Prf}\,(P \vee_I Q) \to \mathrm{Prf}\,(R) \end{aligned}$$

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- a constant for the law of excluded middle:

$$EM : (P : \mathbf{Prop}_C)\mathrm{Prf}\,(P \vee_C \neg_C P)$$

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(\neg\phi)^{\neg\neg} &\equiv \neg\phi^{\neg\neg} \\
(\phi \wedge \psi)^{\neg\neg} &\equiv \phi^{\neg\neg} \wedge \psi^{\neg\neg} \\
(\phi \vee \psi)^{\neg\neg} &\equiv \neg(\neg\phi^{\neg\neg} \wedge \neg\psi^{\neg\neg}) \\
(\phi \to \psi)^{\neg\neg} &\equiv \phi^{\neg\neg} \to \psi^{\neg\neg} \\
(\forall x\phi)^{\neg\neg} &\equiv \forall x\phi^{\neg\neg} \\
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Theorem (Gödel, 1933)

If $PA \vdash \phi$ *then* $HA \vdash \phi^{\neg\neg}$.

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Theorem (Gödel, 1933)

If $PA \vdash \phi$ *then* $HA \vdash \phi^{\neg\neg}$.

How can we make use of this theorem when working in LF?

How It Works

```
class.lf

orC : ...
```

—

Declare the classical system.

How It Works

```
class.lf

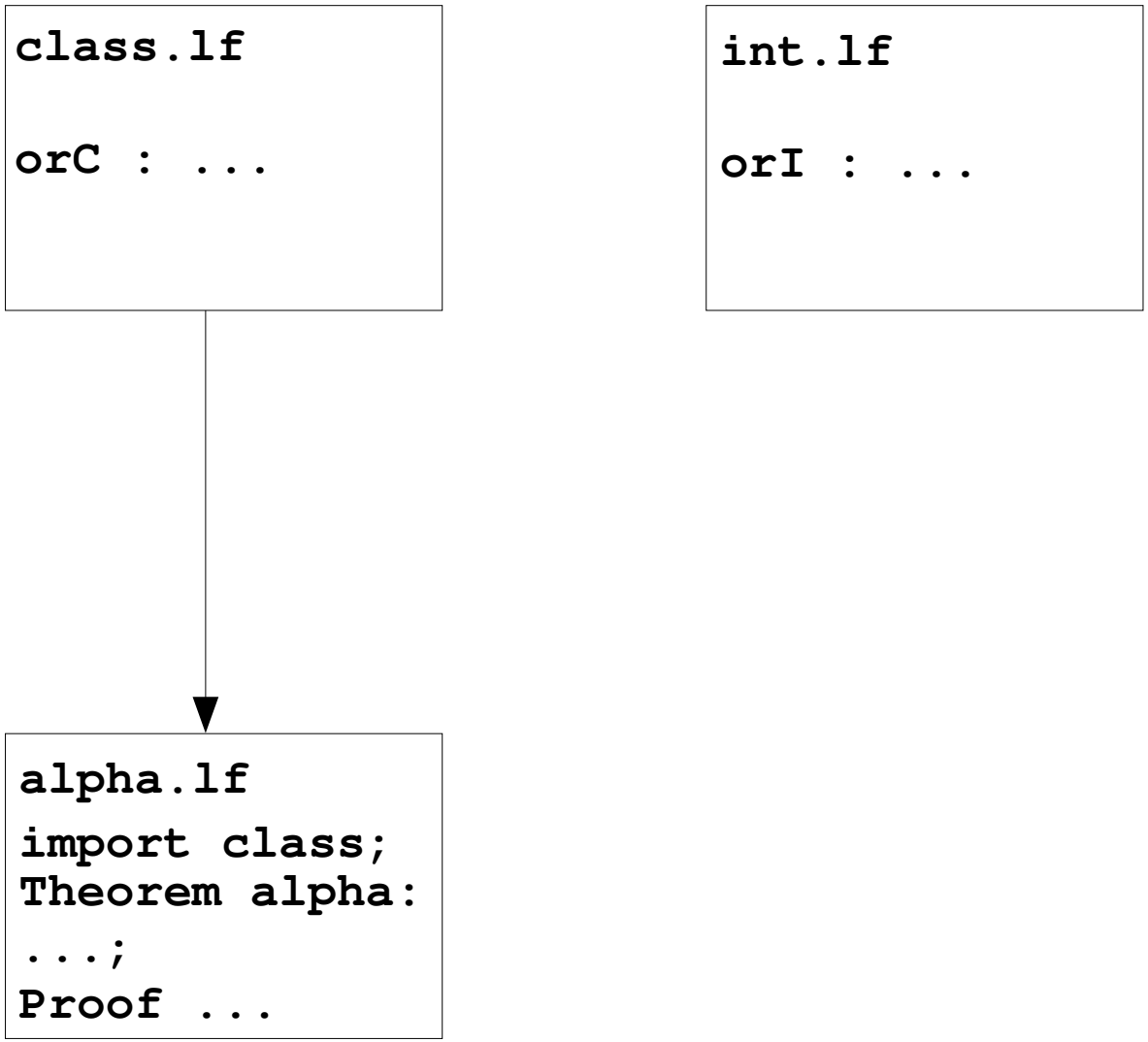
orC : ...
```

```
int.lf

orI : ...
```

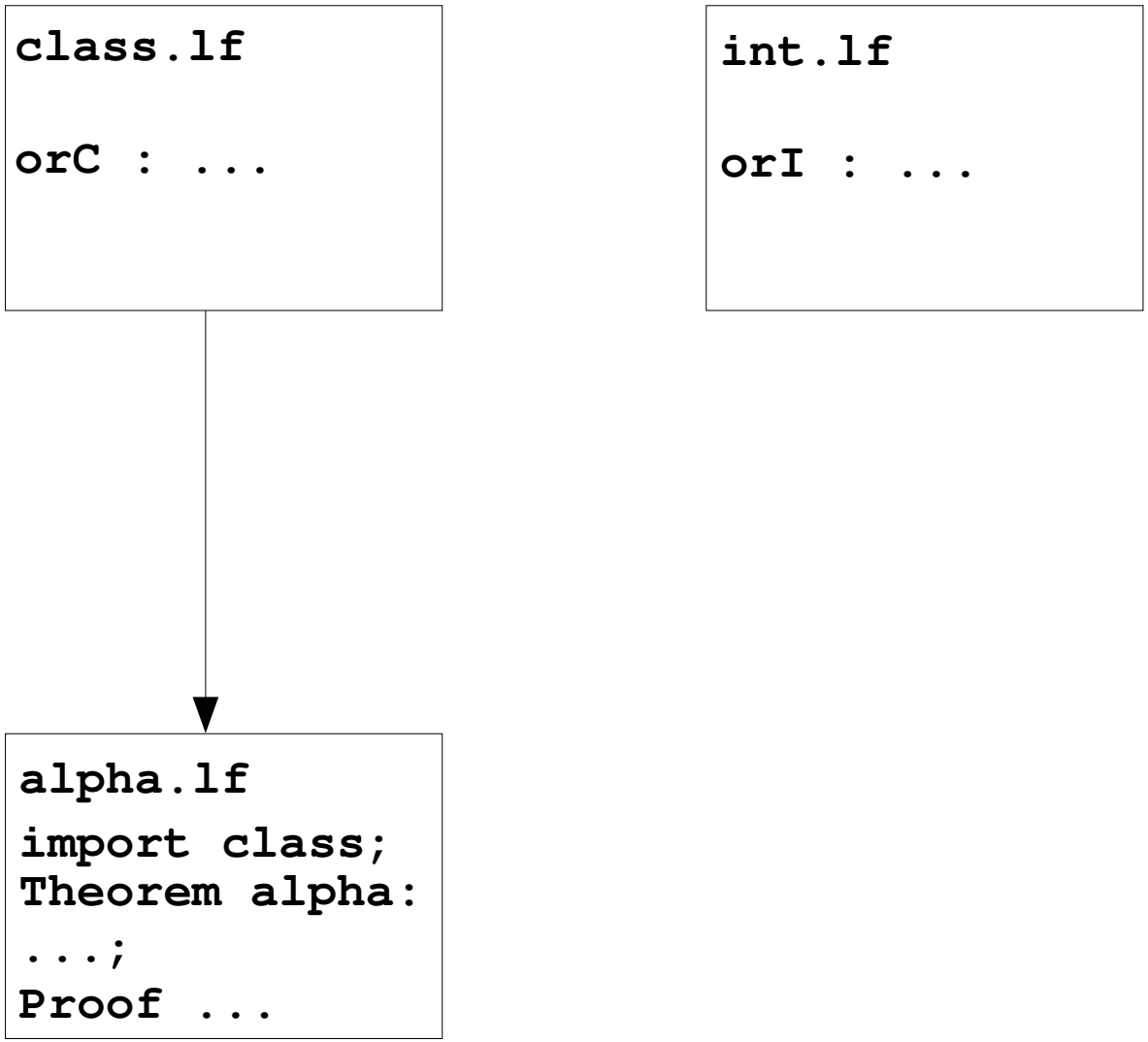
Declare the intuitionistic system.

How It Works



Prove α in the classical system.

How It Works



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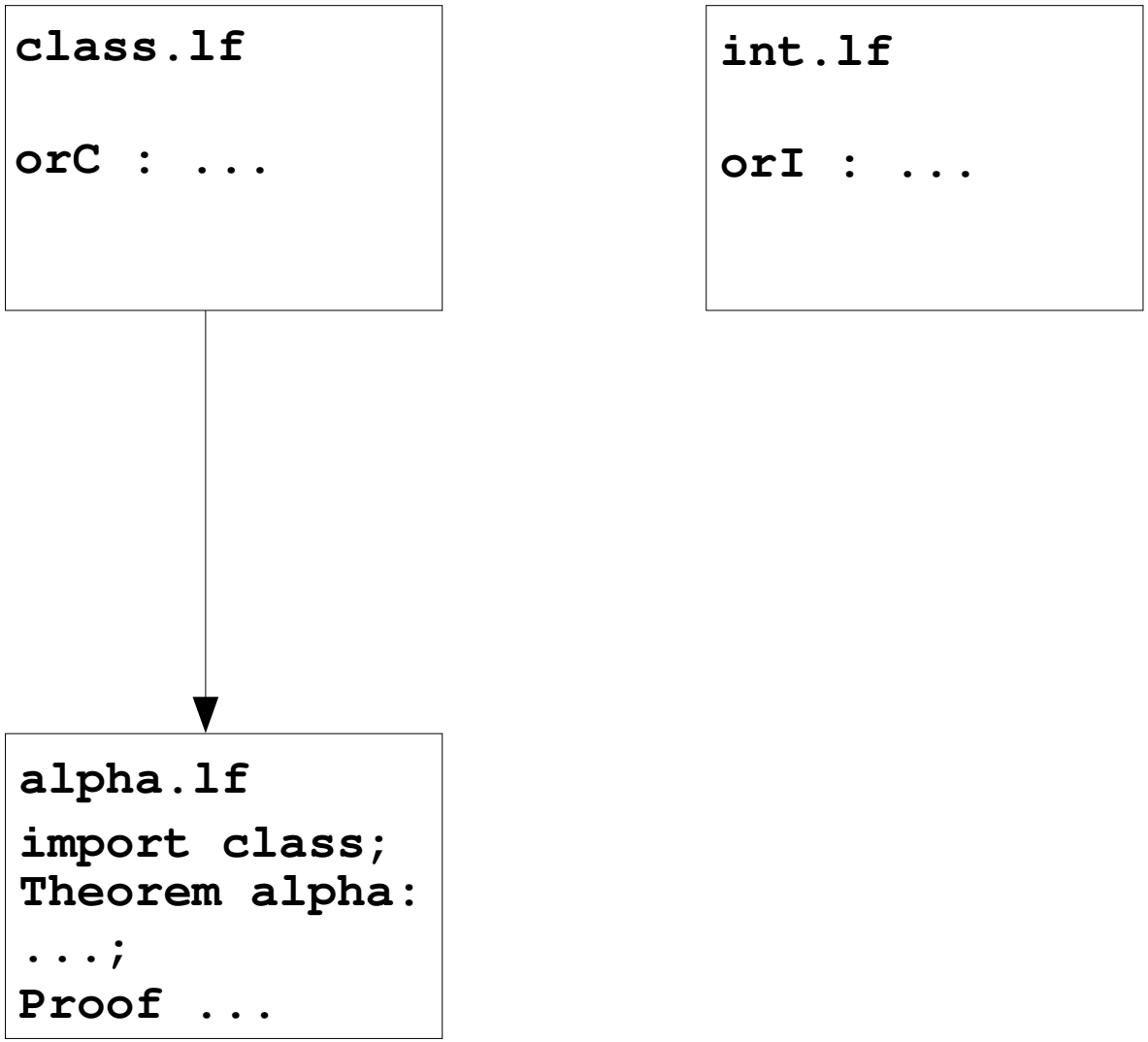

I don't want to do much work now, because I'm lazy.

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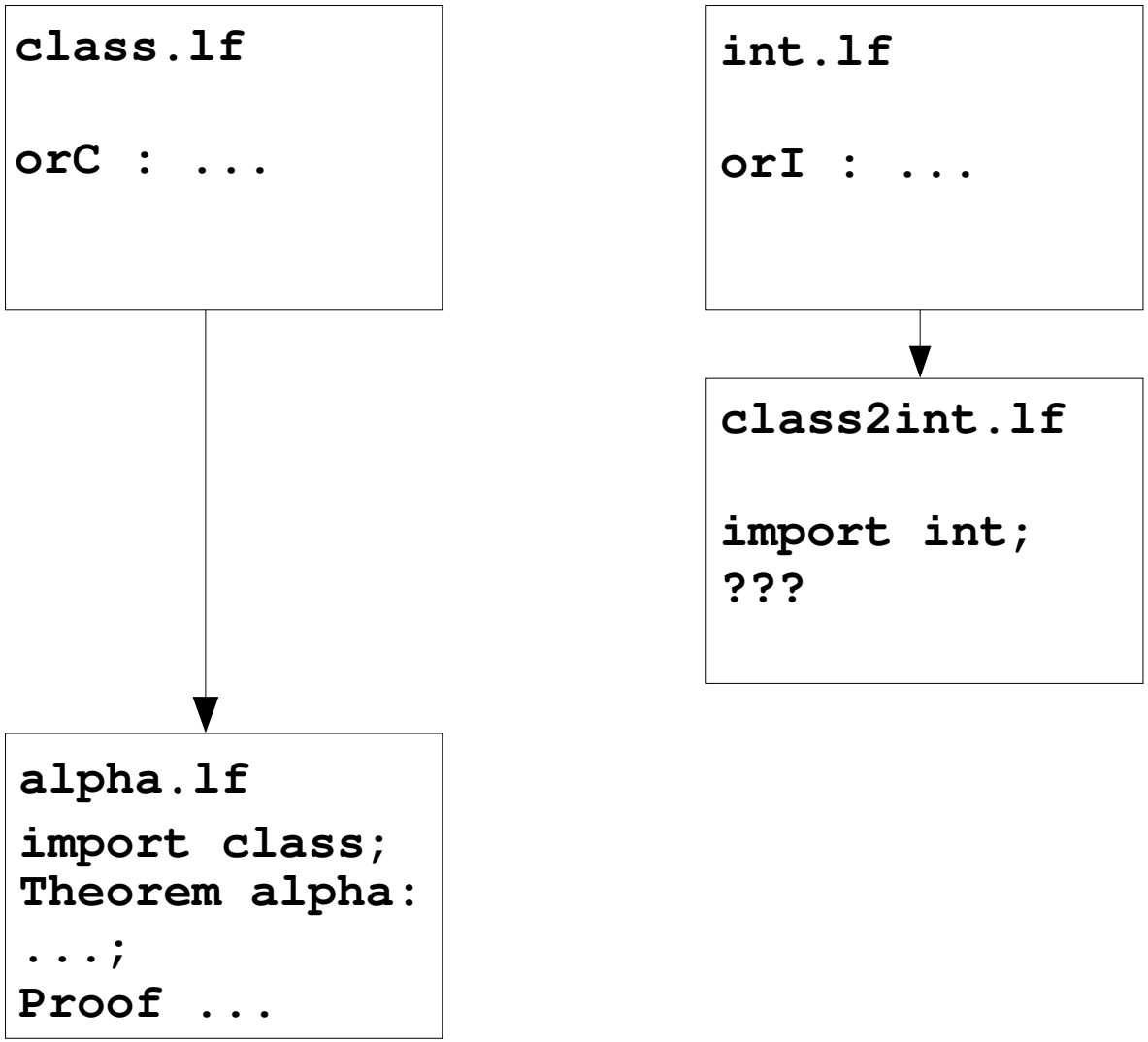
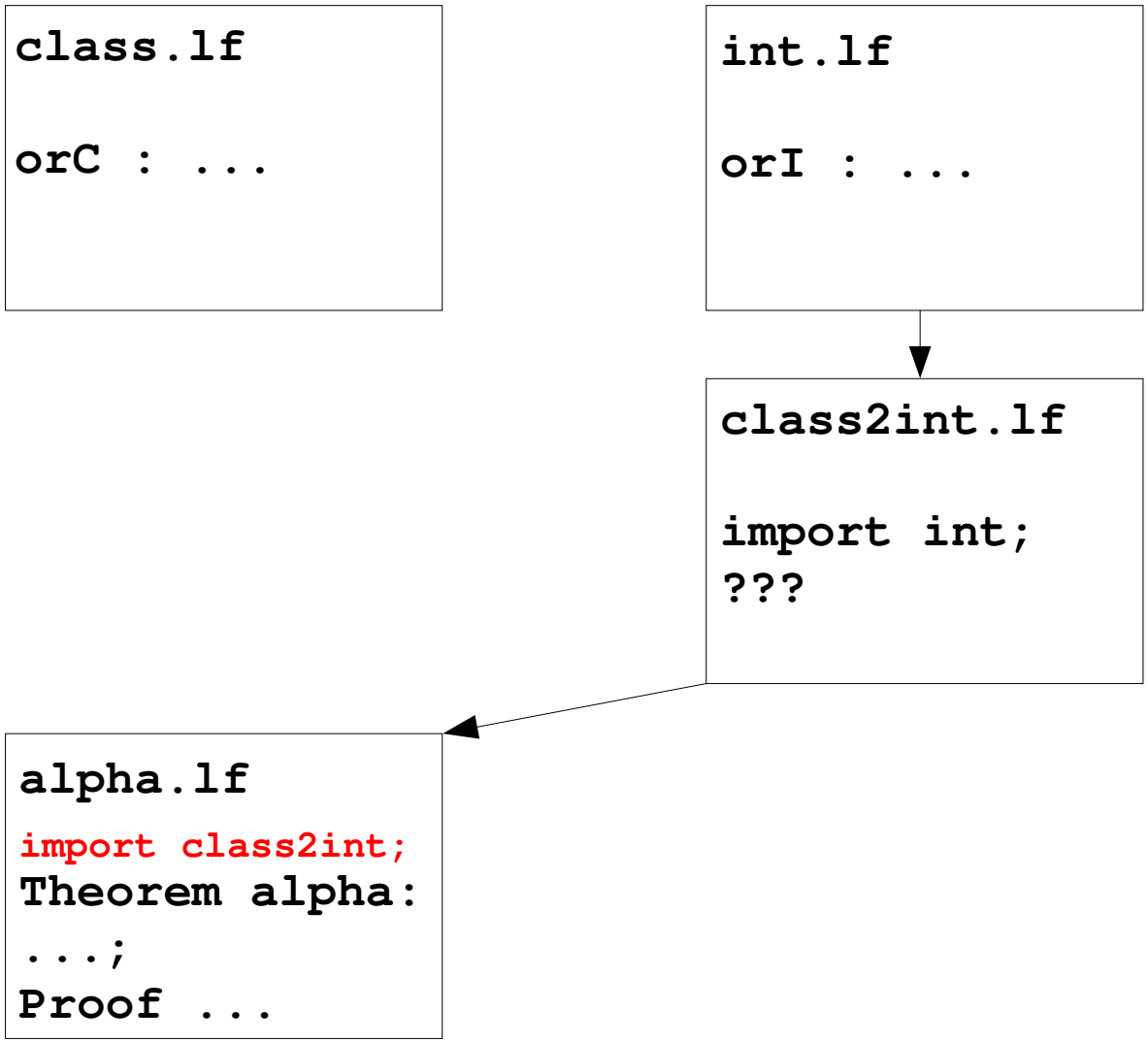


Write a module such that . . .

How It Works



. . . we have a proof of $\alpha^{\neg\neg}$ in the intuitionistic system.

First Attempt

`class.lf`

$$\vdots$$

$$\neg_C \quad : \quad \mathbf{Prop}_C \to \mathbf{Prop}_C$$

$$\vee_C \quad : \quad \mathbf{Prop}_C \to \mathbf{Prop}_C \to \mathbf{Prop}_C$$

$$\exists_C \quad : \quad (\mathbf{Term} \to \mathbf{Prop}_C) \to \mathbf{Prop}_C$$

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`alpha.lf`

```
import class;
```

Theorem alpha : $A \vee_C \neg_C A$

First Attempt

`class2int.lf`

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 & \vdots & \\
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 & : & \mathbf{Prop}_C \to \mathbf{Prop}_C \\
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 & : & \mathbf{Prop}_C \to \mathbf{Prop}_C \to \mathbf{Prop}_C \\
\exists_C & = & [P : \mathbf{Term} \to \mathbf{Prop}_C] \neg_I \forall_I [x : \mathbf{Term}] \neg_I (Px) \\
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`alpha.lf`

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import class2int;
```

Theorem alpha : $A \vee_C \neg_C A \quad \simeq \quad \neg_I(\neg_I A \wedge_I \neg_I \neg_I A)$

The Problem

We need to define an object

$$\begin{aligned} \vee_C E \quad : \quad & (P, Q, R : \mathbf{Prop}) \\ & (\mathrm{Prf}\,(P) \to \mathrm{Prf}\,(R)) \to \\ & (\mathrm{Prf}\,(Q) \to \mathrm{Prf}\,(R)) \to \\ & \mathrm{Prf}\,(P \vee_C Q) \to \mathrm{Prf}\,(R) \end{aligned}$$

The Problem

We need to prove this rule of deduction is derivable:

$$\dfrac{\begin{array}{ccc}[\phi] & [\psi] & \\ \vdots & \vdots & \\ \chi & \chi & \neg(\neg\phi \wedge \neg\psi)\end{array}}{\chi}$$

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. . . but this is not derivable in intuitionistic logic!

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Idea: Define $\mathbf{Prop}_C$ to be the kind of all *stable* formulas.
We would like to write:

$$\mathbf{Prop}_C = \Sigma p : \mathbf{Prop}_I . \neg_I \neg_I p \to p$$

but LF does not have Σ-kinds.

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For now, I used this hack. In `class2int.lf`, declare the constants:

$$\begin{aligned} \mathit{pair} &: (p : \mathbf{Prop}_I)(\neg_I\neg_I p \to p) \to \mathbf{Prop}_C \\ \pi_1 &: \mathbf{Prop}_C \to \mathbf{Prop}_I \\ \pi_2 &: (p : \mathbf{Prop}_C)\neg_I\neg_I \pi_1(p) \to \pi_1(p) \end{aligned}$$

and the computation rule

$$\pi_1(\mathit{pair}\ p\ q) = p : \mathbf{Prop}_I$$

Other Translations and Applications

This method copes with:

- Friedman's A-translation
- The Russell-Prawitz modality

$$\begin{aligned} FOL(\neg, \to, \wedge, \vee, \forall, \exists) &\to SOL(\forall, \to) \\ \text{System T} &\to \text{System F} \end{aligned}$$

It does not quite work with:

- The Dialectica interpretation

$$HA \to \text{System T}$$

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Future Work:

- Implement a module mechanism that makes it more convenient.
- Formalise a piece of pluralist mathematics (e.g. *Metamathematics of First-Order Arithmetic*).
- Use a logical framework to investigate translations between LTTs.