

Formalized foundations of polynomial real analysis

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Purpose of this formalization

Formalize real polynomial analysis **in the SSReflect extension of the Coq proof assistant.**

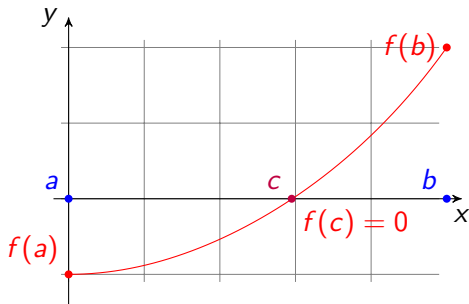
SSReflect provides a lot of **tools** and uses a lot of specific **programming techniques** in the domain of *finite groups* and *combinatorics*.

Reuse these techniques to handle more « continuous » theories.

Real Closed Fields

Algebraic structure of reals : Real Closed Fields (RCF)

Field + Ordered + Intermediate value theorem for polynomials



Decidable Equality

In SSReflect, structures have decidable equality.
We can define this (implicit) coercion in Coq

```
Coercion is_true (b : bool) : Prop := (b = true).
```

SSReflect

- uses intensively this coercion
- has facilities to go from one point of view to the other (bool-Prop reflection).

We then see boolean equality as propositional equality, for free.

What do we need ?

- ⇒ Make case analysis on $x \leq y$
- ⇒ Combine statements (using transitivity with both \leq and $<$, compatibility with operations, etc ..)
- ⇒ Speak about signs and absolute value
- ⇒ Use max and min

Taking advantage of the boolean predicate

Making `le` a boolean predicate.

Like before, consider this boolean predicate as proposition through the coercion `is_true`

⇒ Use equalities to rewrite expressions with order

- $(x+z \leq y+z) = (x \leq y)$
- $(\text{sign } x == 1) = (0 < x)$
- ...

Taking advantage of the boolean predicate

Making `le` a boolean predicate.

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⇒ Use equalities to rewrite expressions with order

- $(x+z \leq y+z) = (x \leq y)$
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- ...

⇒ Use `if x <= y then ... else ...` in programs

- Multiple lemmas about transitivity and compatibility between $\mathbb{1e}$, $\mathbb{1t}$ and field operations
- ⇒ Need for good naming conventions.

Strict comparison

We'll define the strict order lt from the large one le by :

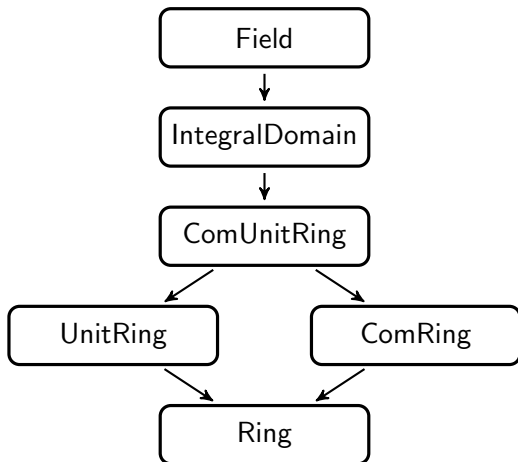
Definition $lt\ x\ y := \sim\sim (le\ y\ x)$.

and prove its properties.

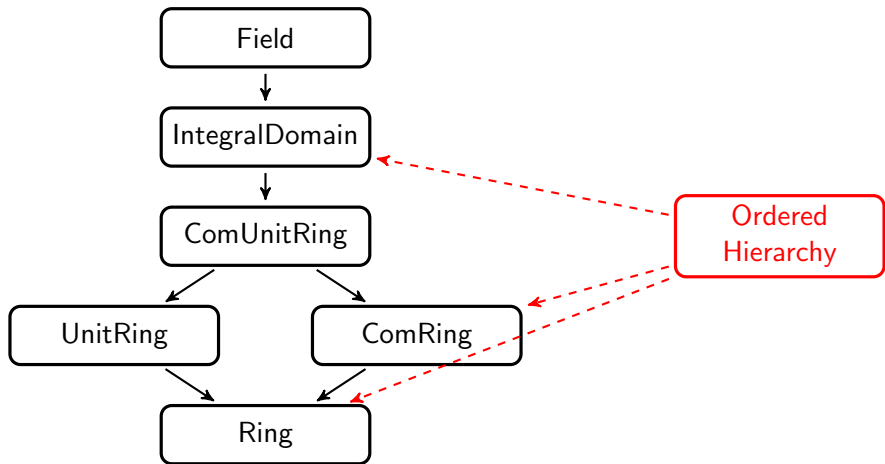
Ordered ring mixin

```
Record mixin_of (R : ringType) := Mixin {  
  le : rel R;  
  _ : antisymmetric le;  
  _ : transitive le;  
  _ : total le;  
  _ : forall z x y, le x y -> le (x + z) (y + z);  
  _ : forall x y, le 0 x -> le 0 y -> le 0 (x * y)  
}.
```

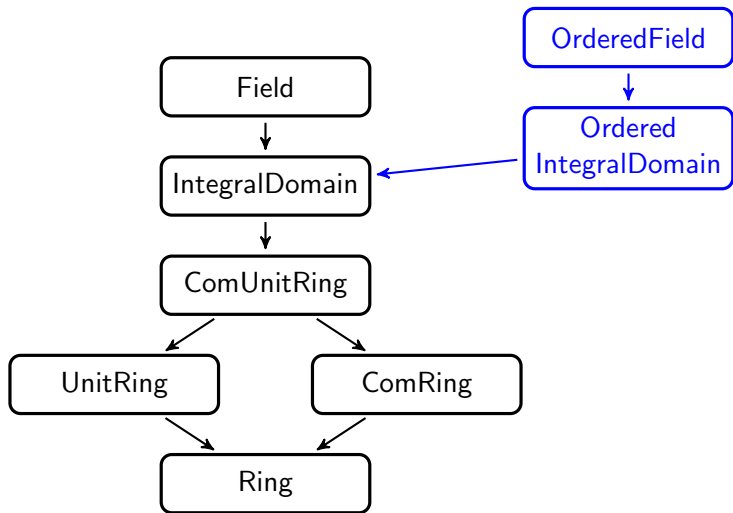
Integration in existing SSReflect algebraic hierarchy



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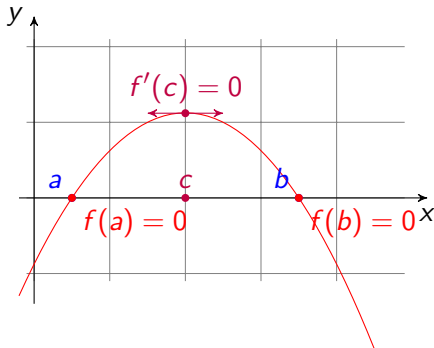


Rolle Theorem for polynomials

A first hint that RCF is a good abstraction of reals :

We are able to prove :

Lemma rolle : forall (a b : R) (p : {poly R}),
a < b -> p.[a] = 0 -> p.[b] = 0 ->
exists c, a < c < b /\ p^'(c).[c] == 0.



Sketch of the constructive proof

Lemma `rolle_weak` : `forall` (a b : R) (p : {poly R}),
a < b -> p.[a] = 0 -> p.[b] = 0 ->
`exists` c , a < c < b
/\ (p^'(c).[c] = 0 \/\ p.[c] = 0).

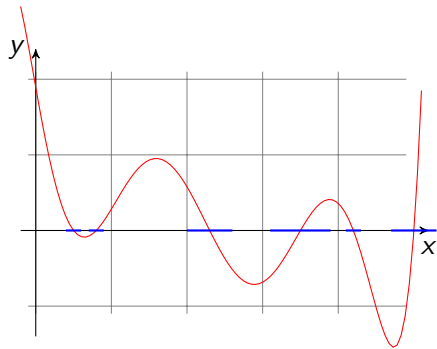
And conclude `rolle` from it by iterating `rolle_weak`. It terminates because P has less than $\deg(P)$ roots.

What else can we do with *IVT*?

Particularly useful examples

- Rolle Theorem
- Mean Value Theorem
- Write a function that computes the real roots of any polynomial
- Prove that given a polynomial P , and a root x of P , one can find a neighborhood of x on which P has no root except x .
- ...

Isolation of roots



Towards quantifier elimination

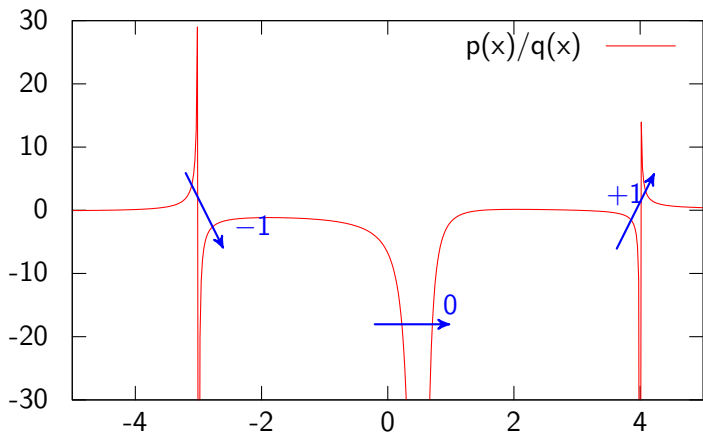
First step of Quantifier Elimination in *RCF*.
Which entails decidability of the theory of *RCF*.

Let's pick one concept from it : Cauchy Index (proof almost done).

Definition of the Cauchy Index

$$\text{CInd}\left(\frac{P}{Q},]a, b[\right) =$$

number of positive jumps – number of negative jumps



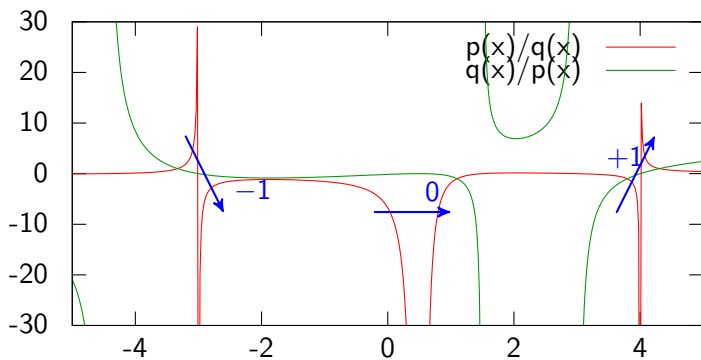
Useful property of Cauchy Index

Property

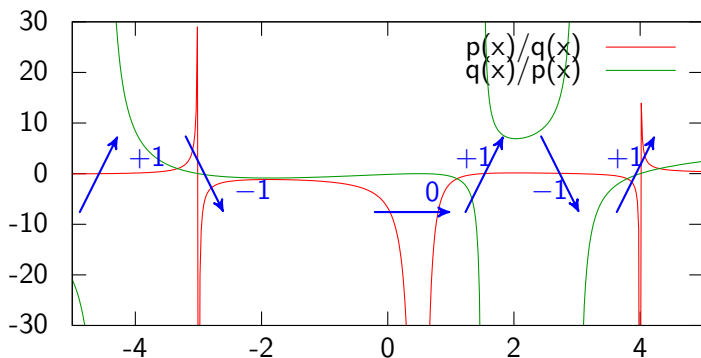
If $P(a), P(b), Q(a), Q(b) \neq 0$ then,

$$\text{CInd} \left(\frac{P}{Q},]a, b[\right) + \text{CInd} \left(\frac{Q}{P},]a, b[\right) = \begin{cases} \text{sign}(PQ(b)) & \text{if } PQ(a)PQ(b) < 0 \\ 0 & \text{else} \end{cases}$$

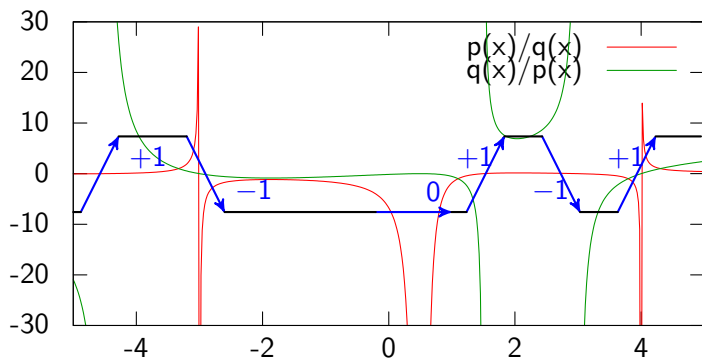
Idea of the proof : combinatorics



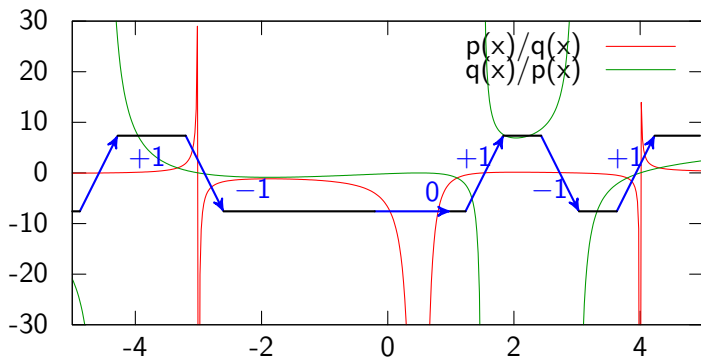
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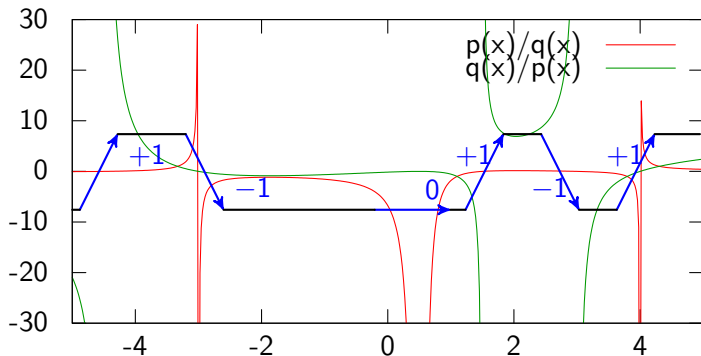
Jumps in the list of *signs* of PQ . $[-1; 1; -1; -1; 1; -1; 1]$

The sum of jumps of a list $I = x_0, \dots, x_n \in \{-1, 1\}^*$ verifies a useful property : it's the jump between x_0 and x_n .

i.e.

$$\begin{cases} \text{sign}(x_n) & \text{if } x_0 x_n < 0 \\ 0 & \text{else} \end{cases}$$

Idea of the proof : combinatorics



Jumps in the list of *signs* of PQ . $[-1; 1; -1; -1; 1; -1; 1]$

Jump between the first sign -1 and the last one 1, i.e.

$$\begin{cases} \text{sign}(PQ(b)) & \text{if } PQ(a)PQ(b) < 0 \\ 0 & \text{else} \end{cases}$$

A library which provides usable tools.

It is used in works in progress on

- Quantifier elimination in *RCF*
- Formalisation of Bernstein Polynomials

What next?

- Instantiate the *Real Closed Fields* Structure
- Prove some reflexive tactics using it
- ... to provide a little more automation
- Generalize notion of continuity in this context
- Extend to further real analysis

The End

Thank you for your attention. Any questions?