Small Scale Reflection, a guided tour

TYPES 2010 - Warsaw 13-16 October 2010

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October 14th 2010

Finite group theory

In all what follows we mainly focus on the study of finite groups.
In all what follows we work in the Coq system.

Finite group theory

- Elements of a group combine thanks to the group law:

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- And most of the theory is developed forgetting about the points.

Decomposition

Theorem (Existence of prime decomposition)

A number always admit a decomposition into a product of prime numbers.

-

Theorem (Existence of composition series)

For any finite group G, there exists a sequence of subgroups:

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Such a sequence is called a composition series for G.
Such a quotient G_{k+1}/G_k is called a factor.

Uniqueness

Theorem (Prime decomposition uniqueness)

The decomposition of any number into a product of primes is unique up to permutations.

-

Theorem (Jordan-Hölder uniqueness)

For any group G, two composition series for G have the same length, and the same factors up to isomorphism and permutation.

-

Proving Jordan Hölder Theorem

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- Base case: $\sharp G = 0$.
  - trivial (a group has at least one element: the neutral)
- Inductive case: $\sharp G > 0$:
  - If G has an empty series: then it is trivial and all its series are empty.
  - If G is simple: then all its series are trivial.
  - Else let (N_i) and (M_j) be two (non empty) composition series of the form.

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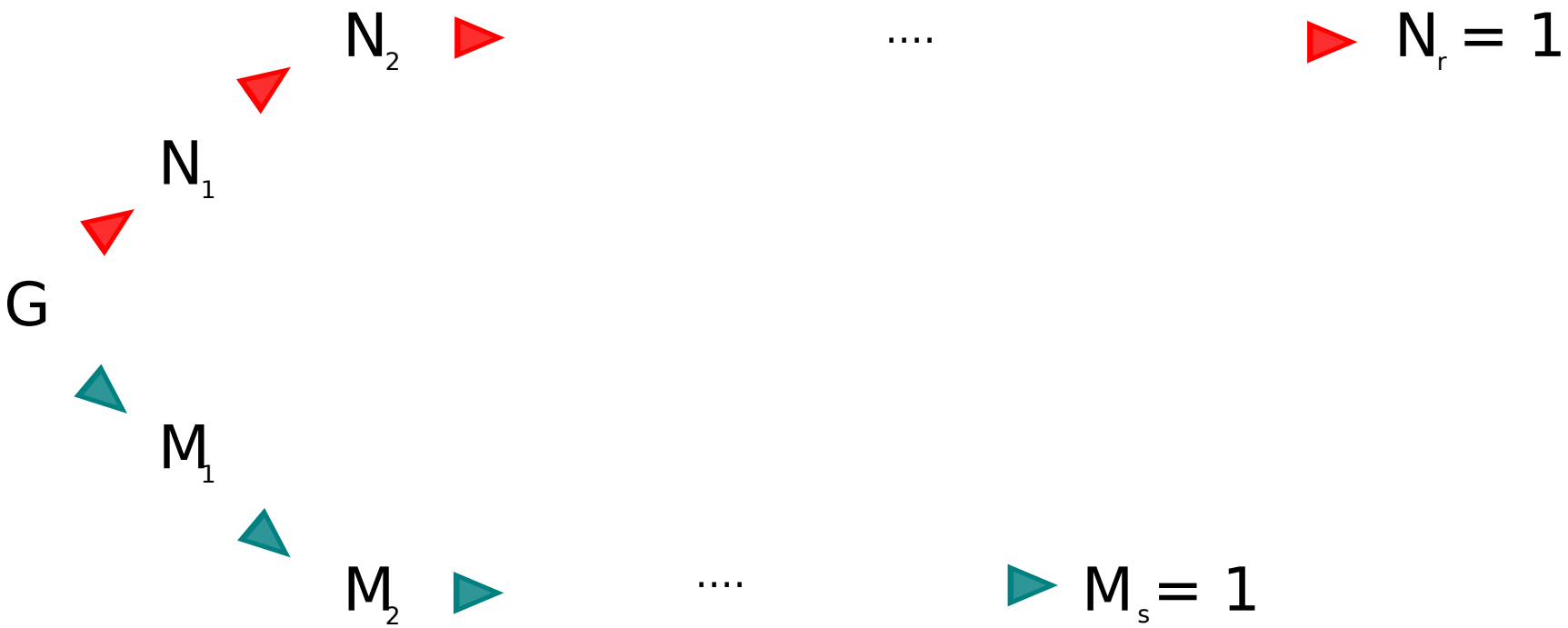

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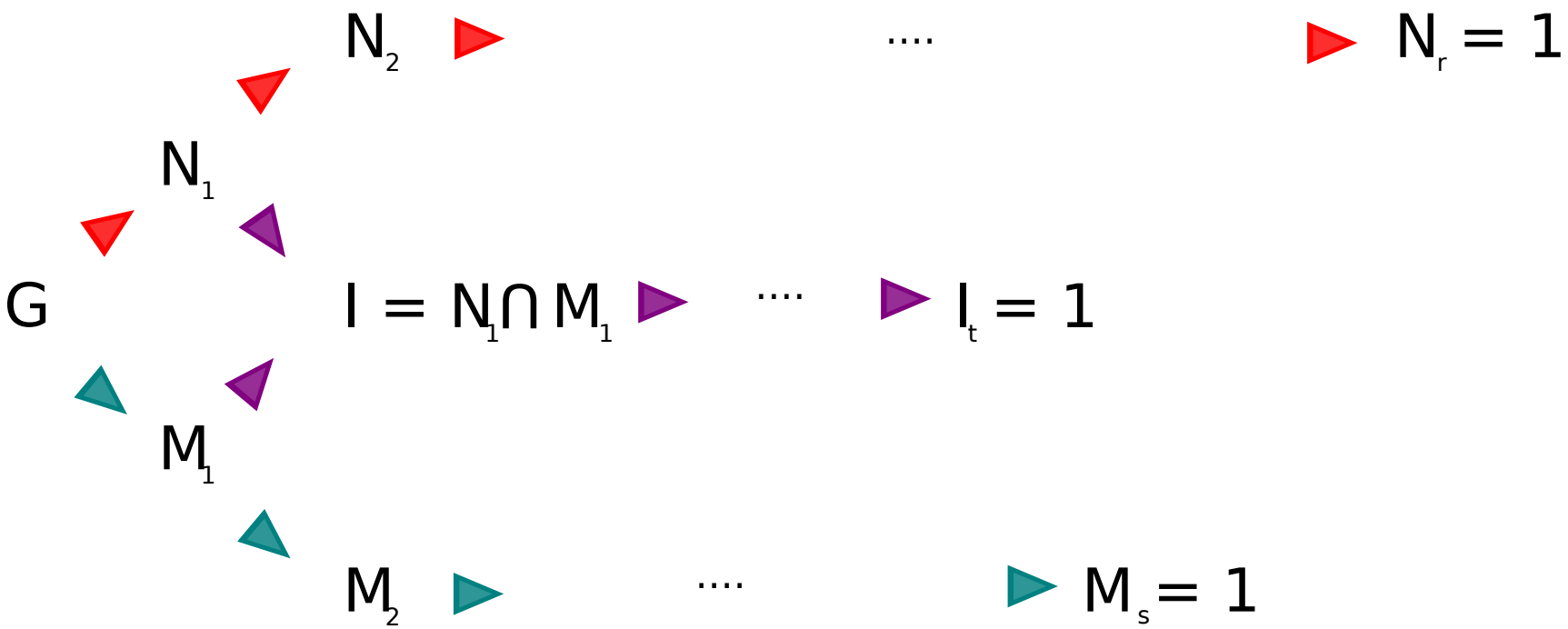

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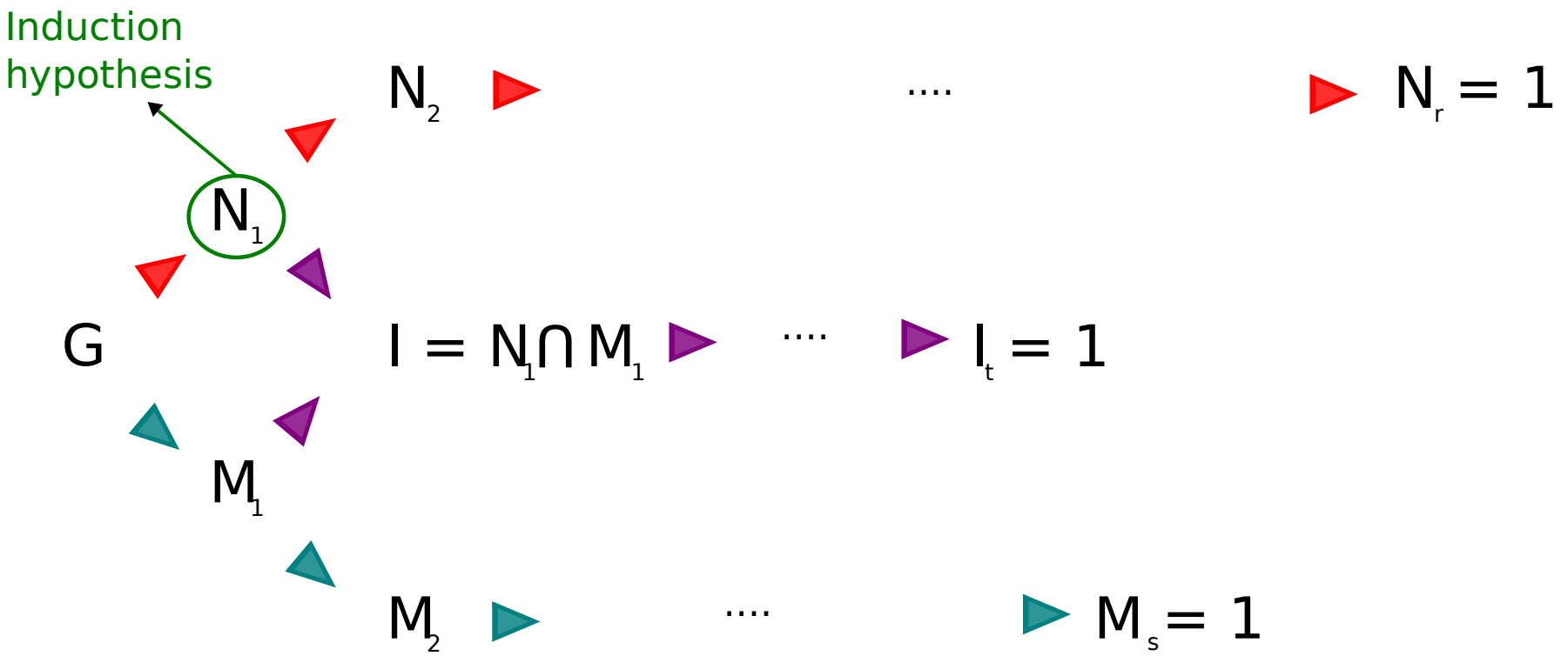

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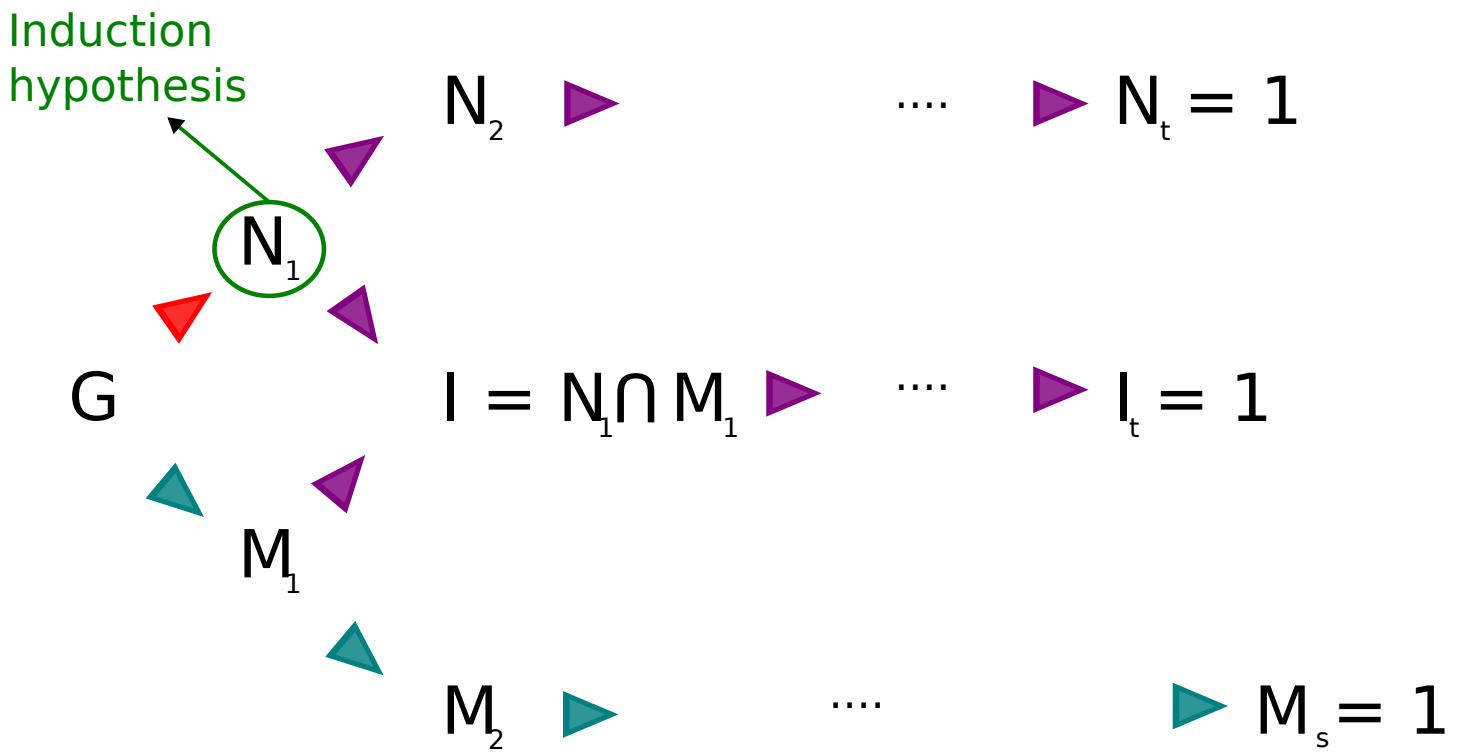

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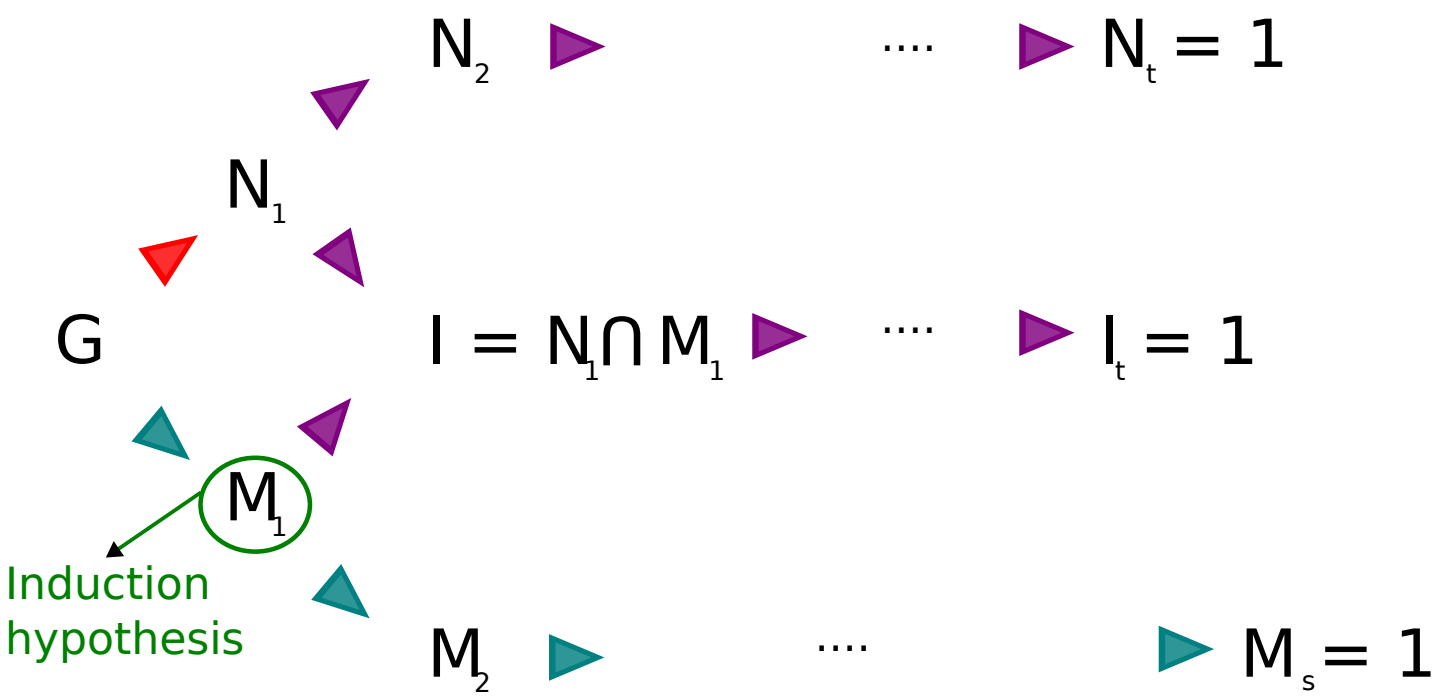

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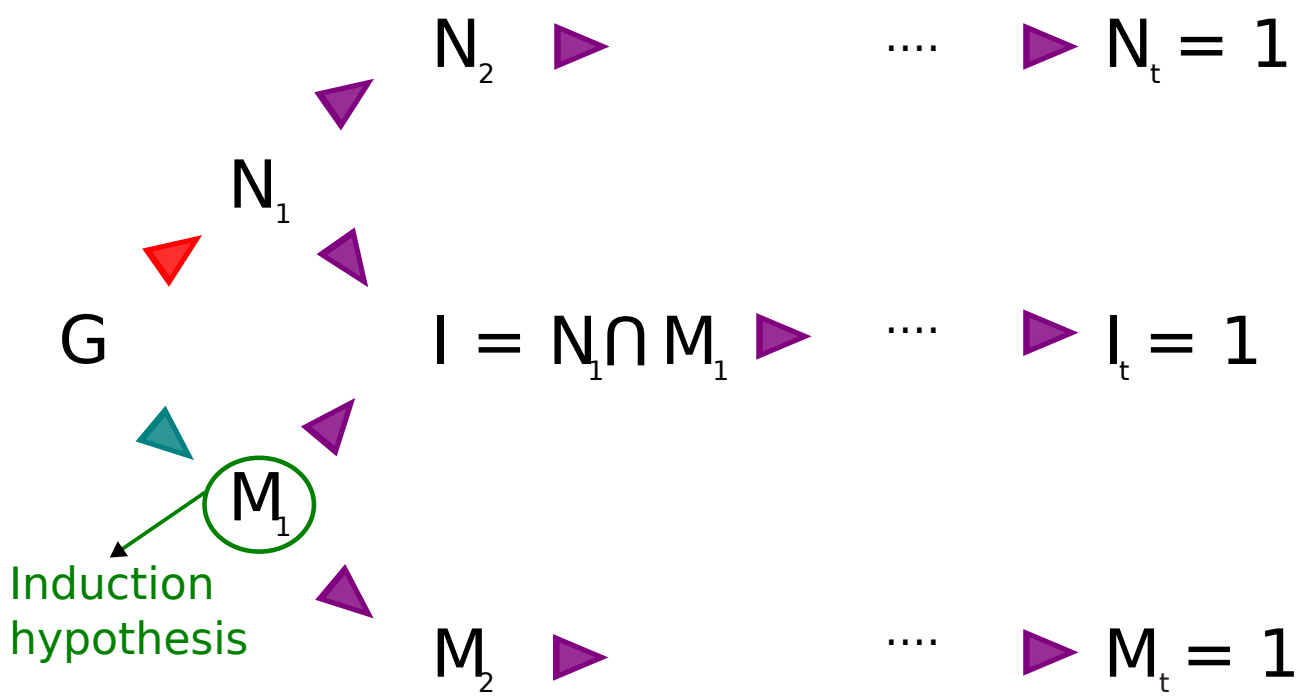

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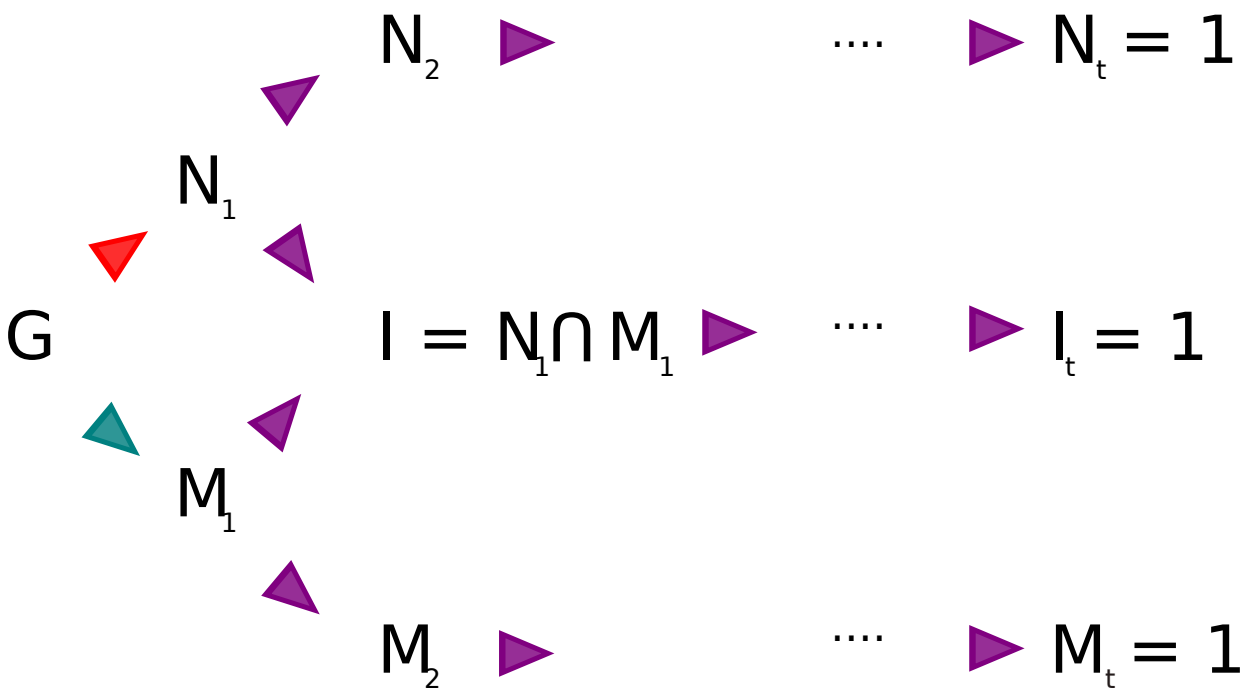

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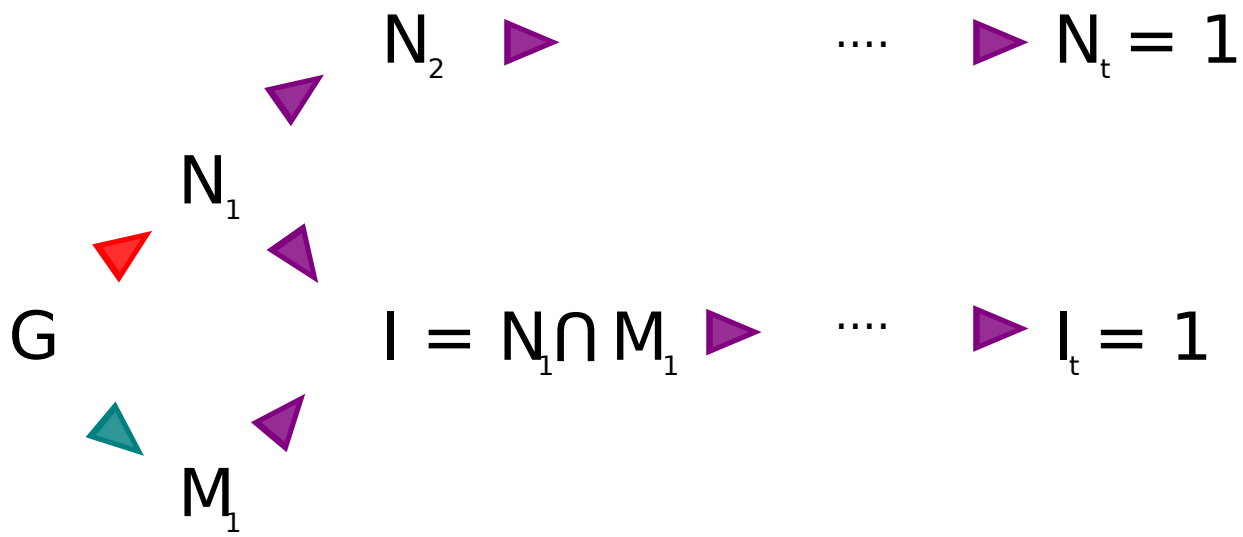

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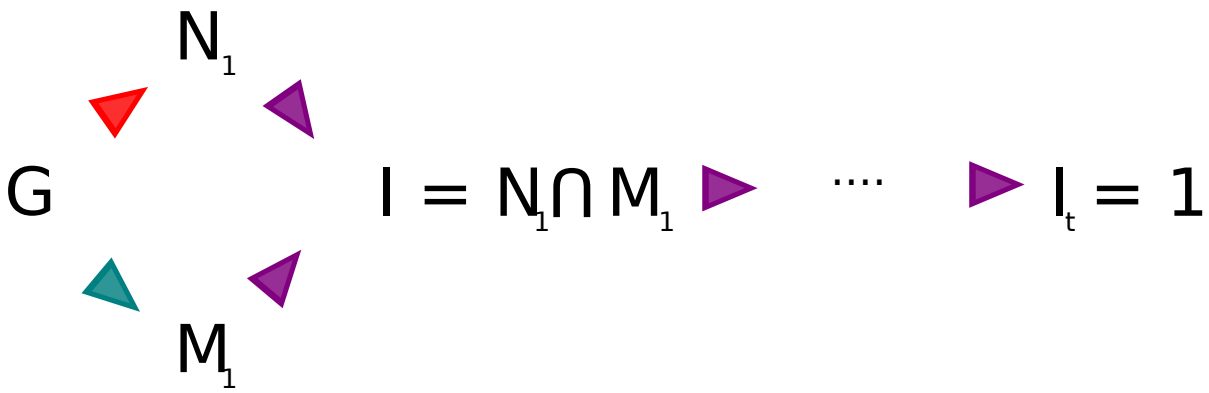
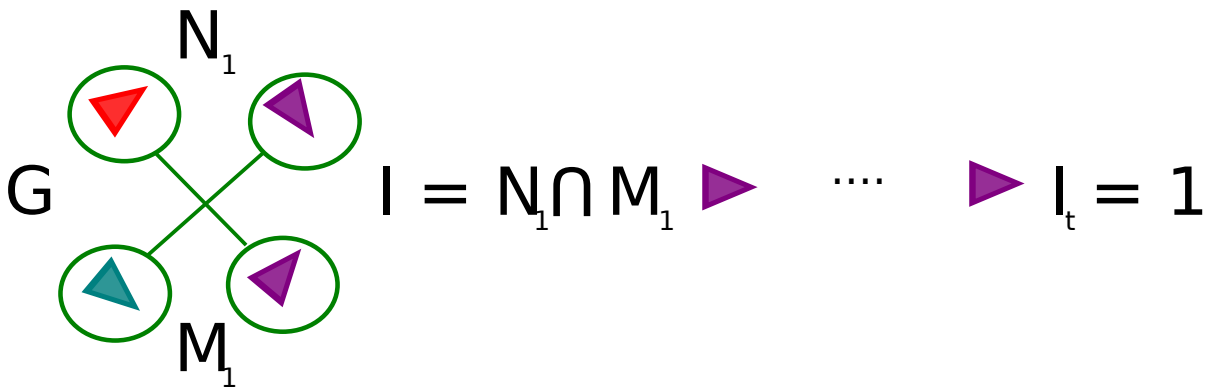


Proving Jordan Hölder Theorem

We conclude by a "butterfly lemma":



Formalization issues: sets vs. types

Not all distinct sets should be modelled as distinct types

- A type fixes common requirements for its inhabitants.
- Type inhabitants are objects we want to observe and combine.

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gT : finGroupType
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  quotient groups...

Formalization issues: comprehension style

- In set theory: comprehension rule forges:
  the set $\{x \mid P\ x\}$
- In type theory: Sigma types (dependent pairs) forge:
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- Is there more to say?

Yes, about the status of equality.

Formalization issues: comprehension style

The sigma type of duplicate free lists on type T is:

$$\{l \ : \ \mathit{list}\, T \mid \text{duplicate\_free } l\}$$

- An inhabitant t_l of this type is a pair (l, p_l)
- Comparing two inhabitants t_1 and t_2 means comparing them component-wise:

$$t_1 = t_2 \quad \text{iff} \quad (l_1 = l_2) \wedge p_{l_1} = p_{l_2}$$

- The proof component should be irrelevant here.

But in general Coq is not a proof irrelevant system...

Formalization issues: comprehension style

A taste of proof-irrelevance:

$$\forall\ (x\ y\ :\ bool)\ (p_1\ p_2\ :\ x = y),\ p_1 = p_2$$

- Suppose that duplicate_free : list T $\rightarrow$ bool

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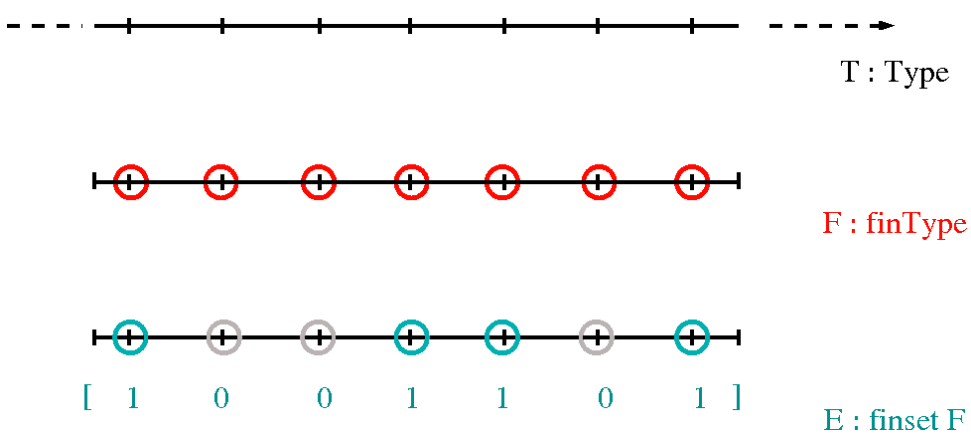
Comparing inhabitants of boolean sigma types is comparing values.

Finite sets are intentional

Ingredients:

- A proof irrelevant definition of lists of fixed length (tuples)
- Types with a finite number of inhabitants

A finite set `(s : set F)` is a mask on a finite type `F`:



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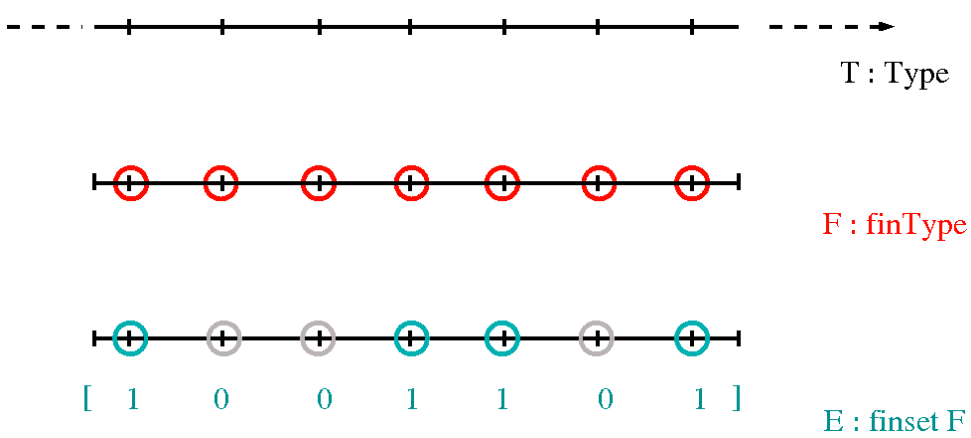

It is a boolean list of fixed length $\sharp$ `F`.

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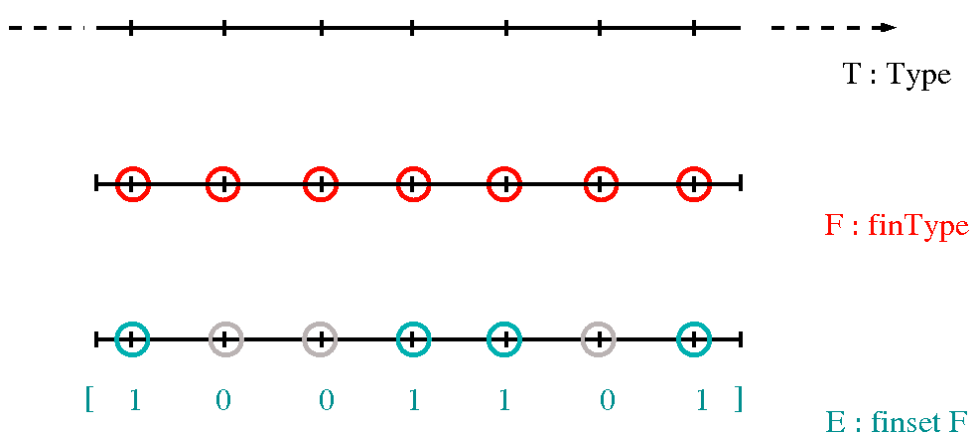

A mask coerces to a characteristic function `(s : F -> bool)`, such that

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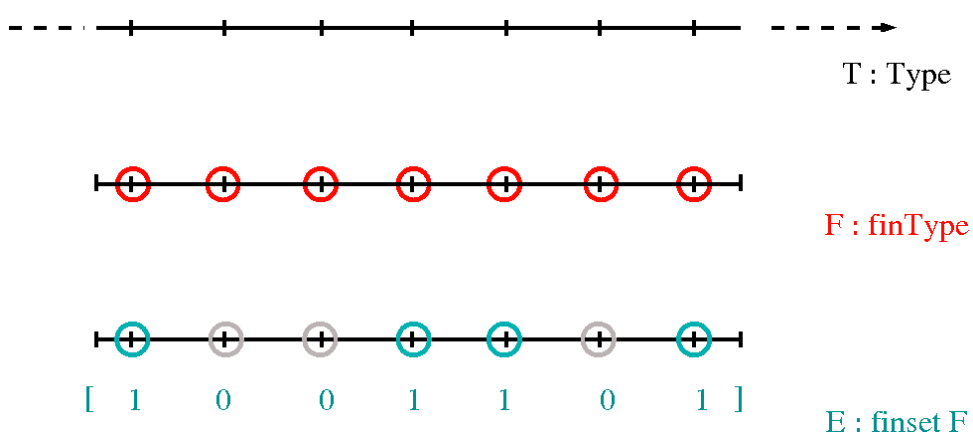

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$\Rightarrow$ Quotient types on finite types are easy.

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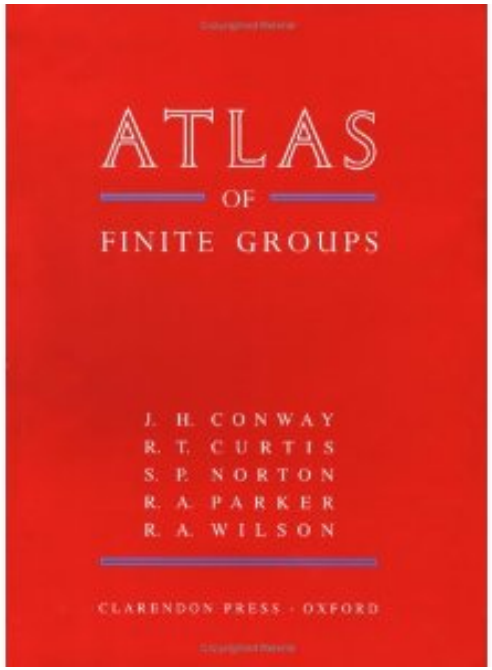
$\Rightarrow$ Classifying finite groups is much more difficult than "classifying" numbers.

The Atlas of Finite Groups

The classification of all simple finite groups
aka. The Enormous Theorem



has been considered achieved in 1983 and revised in 2005.

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- (Wikipedia 12/10/2010) "It takes a professional group theorist about a year of hard work to understand the proof completely."
- The proof mixes many theories, non only combinatorics but also linear algebra, Galois theory, characters,...

Everyday mathematics syntax, implicit semantics

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$$det(M) := \sum_{s \in S_n} (-1)^{\epsilon_s} \prod_i M_{i,s(i)}$$

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In our proof assistant we would like to write:

```
Definition determinant n (A : 'M[R]_n) : R :=
  \sum_(s : 'S_n) (-1) ^+ s * \prod_i A i (s i).
```

Everyday mathematics syntax, implicit semantics

Type inference mechanism play a crucial role here.

In particular, higher order type inference:

- is triggered by higher order unification
- is used as a type classes mechanism, but with proof inference
- even allows quoting (reification by type inference)

The Mathematical Components project

- Who: G. Gonthier, A. Asperti, J. Avigad, Y. Bertot, C. Cohen, F. Garillot, A. M., S. Ould Biha, I. Pasca, L. Rideau, S. Le Roux, S. McLaughlin, R. O'Connor, E. Tassi, L. Théry

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- What is covered as for today:
  - Combinatorics: lists, finite sets, relations, paths, permutations, basic arithmetic, divisibility...
  - Finite group theory: local analysis, fair amount of representation theory
  - Algebra: hierarchy, matrices, linear algebra basics, module theory basics
  - Almost the first volume of the proof

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- What isn't required by this proof: Topology, real, complex analysis, integration, number theory,...

Conclusion

Craft your types with care and benefit from:

- Boolean reflection:
  - relies on the status of computation in CIC
  - requires some formalized infrastructure and support in the tactic language
  - is constructive-compliant
  - can be combined with a classical monad for the possibly classical chunks of a theory
- Higher order type inference:
  - lets implicit in types what is implicit in the textbook
  - makes statements human readable
  - automatizes side conditions inference
- Infrastructure libraries

`http://www.msr-inria.inria.fr/Projects/math-components/`