The Failure of the Range Property for $\mathcal{H}$

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The Problem

Unsolvables

- $\Lambda ::= v \mid \Lambda\Lambda \mid \lambda v \Lambda$
- Hence every $M \in \Lambda$ has one of the following forms:
  - $\lambda x_1 \ldots x_m . y M_1 \ldots M_n$
  - $\lambda x_1 \ldots x_m . (\lambda z . N) M_0 \ldots M_n$

  with $m, n \geq 0$.
- In the first case, M is a *head normal form* (hnf), otherwise it has the *head redex* $(\lambda z.N)M_0$.
- *M has a head normal form* if $M = N$ and N is a hnf.
- $M \in \Lambda^0$ is *solvable* if $\exists \overrightarrow{N} \quad M\overrightarrow{N} = \mathtt{I}$. $M \in \Lambda$ is solvable if the closure of M is solvable. Otherwise, M is *unsolvable*.
- In Scott models, $[\![M]\!] = \bot \iff M$ is unsolvable.
- Wadsworth: M is solvable $\iff M$ has a hnf.
- Barendregt: Unsolvable = Undefined

The Problem (cont.)

The Range Property

- A λ-theory is a congruence on Λ that includes β-convertibility.
- $\mathcal{H}$ is the λ-theory identifying all unsolvable terms. That is, $\mathcal{H}$ is the least congruence containing $=_\beta$ and $\{M = N \mid M, N \text{ unsolvable}\}$.
- If $\mathcal{T}$ is a λ-theory and $F \in \Lambda^0$, then the *range of F in $\mathcal{T}$* is the image of Λ^0 under F, considered as a map from $\Lambda/\mathcal{T}$ to $\Lambda/\mathcal{T}$:

$$\mathrm{Range}^{\mathcal{T}}(F) = \{[FM]_{\mathcal{T}} \mid M \in \Lambda^0\}$$

- A λ-theory $\mathcal{T}$ *has the range property* if

$$\forall F \in \Lambda^0 \quad |\mathrm{Range}^{\mathcal{T}}(F)| \in \{1, \omega\}$$

- The range property is satisfied by λ, $\lambda\eta$, $\mathcal{B}$, $\mathcal{B}\eta$, $\mathcal{H}^*$, all theories induced by standard models of the lambda calculus, and all theories which are computably enumerable.

Strategy

Barendregt [2008] makes the following observations:

Let F be a possible counterexample. [...] Then $x \notin \mathrm{BT}(Fx)$, but $x \in FV(M)$ for all $M =_{\mathcal{H}} Fx$. This means that during the growth of the Böhm tree of M the free variable x is "pushed into infinity". If some trace of x towards infinity occurs in a context $xP_1 \dots P_n$ with n maximal, then the range of F is infinite by considering $F(\lambda x_1 \dots x_n.\mathsf{c}_k)$. The case that is left is that in Fx the free variable x is pushed into infinity and gets more and more arguments to eat. An example of this situation is an F such that $Fx =_\beta \lambda z.z(F(x\Omega)z)$. Then

$$\begin{aligned} Fx &= \lambda z.z(F(x\Omega)z) = \lambda z.z(z((F(x\Omega\Omega)z)) = \dots \\ &= \lambda z.z^n(F(x\Omega^{\sim n})z) = \dots \end{aligned}$$

In this case $\mathrm{Range}^{\mathcal{H}}(F)$ has cardinality 1, as sooner or later $M\Omega^{\sim n} =_{\mathcal{H}} \Omega$. The difficulty is that in general, x, while being pushed to infinity, may get an infinite sequence $P_1, P_2, P_3, \dots$ as arguments (possibly containing the x) and that it is not clear which arguments M can "eat themselves through" this sequence. (We saw that through the sequence $\Omega, \Omega, \cdots$ of cumulative arguments, no M can eat its way, i.e. eventually becomes unsolvable.) It is not decidable which terms can eat themselves through a given infinite sequence.

Example

The F on the previous slide can be generalized.

- Let $\Sigma pxz = z(\Sigma p(xp)z)$. Let $\Sigma^N = \Sigma N$.
- Then $\Sigma^N x = \lambda z.z^k(\Sigma^N(xN^{\sim k})z)$. In particular, $F = \Sigma^\Omega$.
- $\Sigma^{\mathtt{I}} x = \lambda z.z^k(\Sigma^{\mathtt{I}}(x\mathtt{I}\ldots\mathtt{I})z)$.
- For $M, N \in \Lambda^0$, $\Sigma^{\mathtt{I}} M = \Sigma^{\mathtt{I}} N \iff \exists k.\ M\mathtt{I}^{\sim k} = N\mathtt{I}^{\sim k}$
- If $Xp = \lambda z.z(Xp)$, and $X_n = X\mathsf{c}_n$, then

  $$m \neq n \Longrightarrow \Sigma^{\mathtt{I}} X_m \neq \Sigma^{\mathtt{I}} X_n$$

  because $X_m\mathtt{I}^{\sim k} = X_m$, and $X_m \neq X_n$.
- If $N\overrightarrow{P} = \mathtt{I}$, we put $Xp = \lambda z.z\overrightarrow{P}(Xp)$. Then

  $$m \neq n \Longrightarrow \Sigma^N(X\mathsf{c}_m) \neq \Sigma^N(X\mathsf{c}_n)$$

Input Stream Equivalence

Definition

A *tunnel* is a sequence of closed lambda terms.

Let $\langle M_k \rangle = \langle M_0, M_1, \ldots \rangle$ be a tunnel.

- $X\langle M_k \rangle^{\sim n} = X M_0 \cdots M_{n-1}$
- $X \sim_{\langle M_k \rangle} Y \iff \exists n \quad X\langle M_k \rangle^{\sim n} = Y\langle M_k \rangle^{\sim n}$
- X *survives* $\langle M_k \rangle \iff X \nsim_{\langle M_k \rangle} \Omega$
- $\text{Range}(\langle M_k \rangle) = \{[X]_{\sim_{\langle M_k \rangle}} \mid X \in \Lambda^0\}$

We've seen that if $\Omega \neq N \in \Lambda^0$, then

$$M_k = N \Longrightarrow |\text{Range}(\langle M_k \rangle)| = \infty$$

More examples

Which of these tunnels are survivable? Which have infinitely many survivors?

- $\langle M_k \rangle = \langle \mathtt{I}, \Omega, \mathtt{I}, \Omega, \mathtt{I}, \Omega, \cdots \rangle$
  - $X = \lambda zy.zX$ survives M_k
  - $Xp = \lambda zy.z(Xp)$ gives infinity survivors.
- $\langle M_k \rangle = \langle \lambda x.x, \lambda x.x\Omega, \lambda x.x\Omega\Omega, \cdots \rangle$
  - $X = (\lambda uz.zu(X(\mathtt{K}u))\mathtt{I}$
  - $Xp = (\lambda uz.zu(Xp(\mathtt{K}u))\mathtt{I}$ gives infinity survivors.
- $\langle M_k \rangle = \langle \mathtt{I}, \Omega, \mathtt{I}, \Omega, \Omega, \mathtt{I}, \Omega, \Omega, \Omega, \mathtt{I}, \cdots \rangle$
  - $X = (\lambda uz.zu(X(\lambda z.u(\mathtt{K}z))))\mathtt{I}$
  - $Xp = \ldots$
- $\langle M_k \rangle = \langle \lambda x.x, \Omega, \lambda x.x\Omega, \Omega, \Omega, \lambda x.x\Omega\Omega, \cdots \rangle$
  - $X = (\lambda uz.z(u\mathtt{I})u(X(\lambda z.u(\mathtt{K}z))))\mathtt{I}$

Conclusion: The range of $\langle M_k \rangle$ is infinite if we can "hop over" the unsolvable terms while carrying some parameter.

The computable case

The previous statement can be made rigorous:

Proposition

If $\langle M_k \rangle$ *is closed and computable, then* $|\mathrm{Range}\langle M_k \rangle| \in \{1, \infty\}$

We can do better.
Say that a term M is *fully solvable* if for some $\overrightarrow{N}$, $M\overrightarrow{N} = \mathtt{I}$.

Theorem

If infinitely many fully solvable terms in $\langle M_k \rangle$ *can be computably enumerated, then* $\mathrm{Range}(\langle M_k \rangle) = \infty$.

So if M has finite range in $\mathcal{H}$, then every trace of x must get an infinite applicative context with no closed, computably enumerable subsequence. Converse?

The Devil's tunnel

Let $f : \mathbb{N} \to \mathbb{N}$ be strictly monotone with $f(0) = 0$.

- We think of f as giving the *islands of survivability* in the sequence $\langle M_n \rangle$.
- Let $\triangle f(n) = f(n+1) - f(n) - 1$.
- Recall that $\mathtt{U}_k^k = \lambda x_0 \ldots x_k . x_k$, then $\mathtt{U}_k^k N_0 \ldots N_{k-1} = \lambda x_k . x_k = \mathtt{I}$.
- The *Devil's tunnel of f* is defined as follows:

$$F_n := \begin{cases} \Omega & n \notin \mathrm{Range}(f) \\ \mathtt{U}_{\triangle f(k)}^{\triangle f(k)} & n = f(k) \end{cases}$$

- For any f, n, $\mathtt{I}\langle F_k \rangle^{\sim f(n)} = \mathtt{I}$. Hence $\mathtt{I} \not\sim_{\langle F_k \rangle} \Omega$.

Example

Let $\langle f(k)\rangle = \langle 0, 2, 7, 8, 235, \ldots\rangle$

Then $\langle \triangle f(k)\rangle = \langle 1, 4, 0, 226, \ldots\rangle$, and

$$\begin{aligned}
\mathtt{I}\langle F_k\rangle^{\sim f(5)} &= \mathtt{I}\mathtt{U}_1^1\Omega\mathtt{U}_4^4\Omega\Omega\Omega\Omega\mathtt{U}_0^0\mathtt{U}_{226}^{226}\overbrace{\Omega\cdots\Omega}^{226} \\
&= \mathtt{U}_1^1\Omega\mathtt{U}_4^4\Omega\Omega\Omega\Omega\mathtt{U}_0^0\mathtt{U}_{226}^{226}\Omega\cdots\Omega \\
&= \mathtt{U}_4^4\Omega\Omega\Omega\Omega\mathtt{U}_0^0\mathtt{U}_{226}^{226}\Omega\cdots\Omega \\
&= \mathtt{U}_0^0\mathtt{U}_{226}^{226}\Omega\cdots\Omega \\
&= \mathtt{U}_{226}^{226}\Omega^{\sim 226} \\
&= \mathtt{U}_0^0 = \mathtt{I}
\end{aligned}$$

Generally,

$$\mathtt{I}F_0F_1\cdots F_{f(n)}\cdots = \mathtt{I}\ \underline{\mathtt{U}_1^1\Omega}\ \underline{\mathtt{U}_4^4\Omega\Omega\Omega\Omega}\ \underline{\mathtt{U}_0^0}\ \underline{\mathtt{U}_{226}^{226}\Omega\cdots\Omega^{226}}\cdots$$

Main lemma

Definition

A *tree* is a partial map $T : \mathbb{N}^* \rightharpoonup \mathbb{N}$ such that

- $\sigma \in \mathrm{dom}(T), \tau \subseteq \sigma \implies \tau \in \mathrm{dom}(T)$
- $\sigma * \langle i \rangle \in \mathrm{dom}(T) \implies i \leq T(\sigma)$

We have seen that $\mathrm{Range}(\langle F_k \rangle)$ has at least two elements.

Theorem

Let $\langle F_k \rangle$ be the Devil's tunnel of f.

$$|\mathrm{Range}(\langle F_k \rangle)| > 2 \iff \Delta f \text{ is a path through a c.e. tree}$$

Intuition:

$$\begin{aligned} & M = (\lambda \vec{x}.x_i P_1 \dots P_p)\mathrm{U}_{k_1}^{k_1}\Omega \cdots \Omega \mathrm{U}_{k_2}^{k_2}\Omega \cdots \text{ solvable} \\ \implies \quad & M = (\mathrm{U}_k^k P_1 \dots P_p) \dots \mathrm{U}_{k'}^{k'}\Omega \dots \\ & \quad = (P_{k+1} \dots P_p) \dots \\ & \quad = (\lambda \vec{y}.y_{i'} P'_1 \dots P'_{p'}) \dots \end{aligned}$$

Consequence

Proposition

There exists a function $f : \mathbb{N}^* \to \mathbb{N}$ that is not a path through a c.e. tree.

Proof.

Finite injury. □

Theorem

The range property fails for $\mathcal{H}$.

Proof.

The construction can be done within a λ-term. □

The Counterexample

Notation

- For $M \in \Lambda^0$, $\#M$ is the *Gödel number* of M, and $\ulcorner M \urcorner = \mathsf{c}_{\#M}$ is the *quote* of M.
- For $\sigma \in 2^*$, $\#\sigma$ is the Gödel number of σ, and $\ulcorner \sigma \urcorner = \mathsf{c}_{\#\sigma}$ is the quote of σ.
- $|\sigma|$ is the length of σ.
- $\mathtt{E}$ is an evaluator ($\forall M \quad \mathtt{E}\ulcorner M \urcorner = M$).
- We put $L_n = \mathtt{E}\mathsf{c}_n$, so that $\Lambda^0 = \{L_0, L_1, \dots\}$.
- We write $M{\downarrow}^k$ if the head reduction of M terminates after at most k steps.
- $M{\downarrow}$ is $\exists k.\ M{\downarrow}^k$. Otherwise, $M{\uparrow}$.
- If $M{\downarrow}$, then $\mathsf{phnf}(M)$ is the principal head normal form of M.

$$\|M\| = \begin{cases} l + m & \mathsf{phnf}(M) = \lambda x_0 \dots x_l . y P_1 \cdots P_m, \\ \uparrow & M \text{ unsolvable} \end{cases}$$

The Counterexample

Definition

We define a family of terms $\{M^\sigma\}_{\sigma \in 2^*} \subseteq \Lambda(x)$ by induction on σ:

$$
\begin{array}{rcl}
M^{\langle\rangle} & = & x \\
M^{\sigma * \langle 0 \rangle} & = & M^\sigma \mathtt{I} \\
M^{\sigma * \langle 1 \rangle} & = & \begin{cases} M^\sigma \mathtt{U}^k_k \Omega^{\sim k} & k = \| M^\sigma [x := L_{|\sigma|}] \| \\ \Omega & M^\sigma [x := L_{|\sigma|}] \uparrow \end{cases}
\end{array}
$$

Let $M^\sigma_\natural = M^\sigma [x := L_{|\sigma|}]$.

Definition

We go over to the codes; let F λ-define the following pcf f:

$$
\begin{array}{rcl}
f(\#\langle\rangle) & = & \#\mathtt{I} \\
f(\#(\sigma * \langle 0 \rangle)) & = & \#\lambda z. L_{f(\#\sigma)} z \mathtt{I} \\
f(\#(\sigma * \langle 1 \rangle)) & = & \begin{cases} \#\lambda z. L_{f(\#\sigma)} z \mathtt{U}^k_k \Omega^{\sim k} & k = \| L_{f(\#\sigma)} L_{|\sigma|} \| \\ \uparrow & L_{f(\#\sigma)} L_{|\sigma|} \uparrow \end{cases}
\end{array}
$$

The Counterexample

The Term

$$S\ulcorner\sigma\urcorner = \begin{cases} \lambda z.z\mho_k^k\Omega^{\sim k} & k = \|M_\natural^\sigma\| \\ \Omega & M_\natural^\sigma\uparrow \end{cases}$$

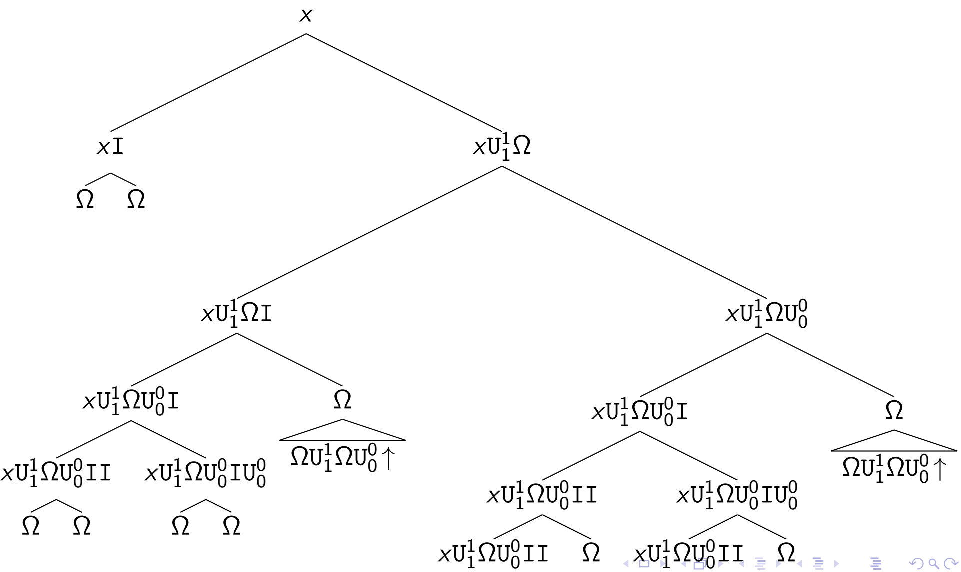
$$T\ulcorner\sigma\urcorner = \begin{cases} \Omega & M^\sigma\uparrow \text{ or } \exists\ \tau * \langle 0\rangle \subseteq \sigma.\ M_\natural^\tau\downarrow^{|\sigma|}, \\ \mathtt{I} & \text{otherwise} \end{cases}$$

$$W\ulcorner\sigma\urcorner x = T\ulcorner\sigma\urcorner[W\ulcorner\sigma * \langle 0\rangle\urcorner(x\mathtt{I}),\ W\ulcorner\sigma * \langle 1\rangle\urcorner(S\ulcorner\sigma\urcorner x)]$$

$$\Xi = W\ulcorner\langle\rangle\urcorner \qquad (=_\eta \lambda x.W\mathsf{c}_0 x)$$

The Counterexample

Suppose that $L_0, L_1, L_2, L_3, L_4 \cdots = \mathtt{K}, \omega_3, \Omega, \mathtt{I}, \mathtt{KI}, \ldots$.



The Counterexample

Let σ be a position.

σ is *doomed* if $\exists \tau * \langle 0 \rangle \subseteq \sigma \quad M_\natural^\tau\downarrow$.

σ is *healthy* if it is not doomed and $M^\sigma\downarrow$.

- $\forall n \exists! \sigma \quad |\sigma| = n$ and σ is healthy.

  Base case: $|\sigma| = 0 \implies \sigma = \langle\rangle$

  Induction: Suppose σ is healthy. We have
  $M_\natural^\sigma\uparrow \implies M^{\sigma * \langle 1 \rangle}\uparrow, \Xi_{\sigma * \langle 0 \rangle}\downarrow$ keeps $\sigma * \langle 0 \rangle$ healthy.
  At the same time, $M_\natural^\sigma\downarrow^k \implies \sigma * \langle 0 \rangle$ is doomed,
  while $\Xi_{\sigma * \langle 1 \rangle}$ is OK.
- $x \notin \Xi_\sigma$ unless σ is healthy.
- $\Xi_\sigma x = W \ulcorner \sigma \urcorner M^\sigma$.
- $\Xi X = \Xi \Omega \iff \exists n.\ M^{\sigma_n}[x := X]\uparrow$

The Counterexample

Punchline: $\Xi X \neq \Xi\Omega \Longrightarrow \Xi X = \Xi \mathtt{I}$

- Let σ_n be the healthy σ of length n.
- Let $X = \mathtt{Ec}_e$ be given. Let $\sigma = \sigma_e$. Let $k = \|X\|$.
- Then

$$\begin{aligned}
\Xi_\sigma X &= X^{\ulcorner}\sigma^{\urcorner} M^\sigma[x := X] \\
&= W^{\ulcorner}\sigma^{\urcorner} M^\sigma_\natural \neq X^{\ulcorner}\sigma^{\urcorner}\Omega \\
&= [W^{\ulcorner}\sigma * \langle 0\rangle^{\urcorner}(M^\sigma \mathtt{I})[x := X], W^{\ulcorner}\sigma * \langle 1\rangle^{\urcorner}(M^\sigma_\natural \mathtt{U}^k_k \Omega^{\sim k})] \\
&= [W^{\ulcorner}\sigma * \langle 0\rangle^{\urcorner}\Omega, W^{\ulcorner}\sigma * \langle 1\rangle^{\urcorner} N]
\end{aligned}$$

If $M^\sigma_\natural = \lambda x_0 \ldots x_l . x_i P_1 \ldots P_m$, $l + m = k$, then either $N = \Omega$ or $i = 0$ and $N = \mathtt{U}^k_k P_1 \ldots P_m \Omega^{k-l} = \mathtt{U}^l_l \Omega^{\sim m}$

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