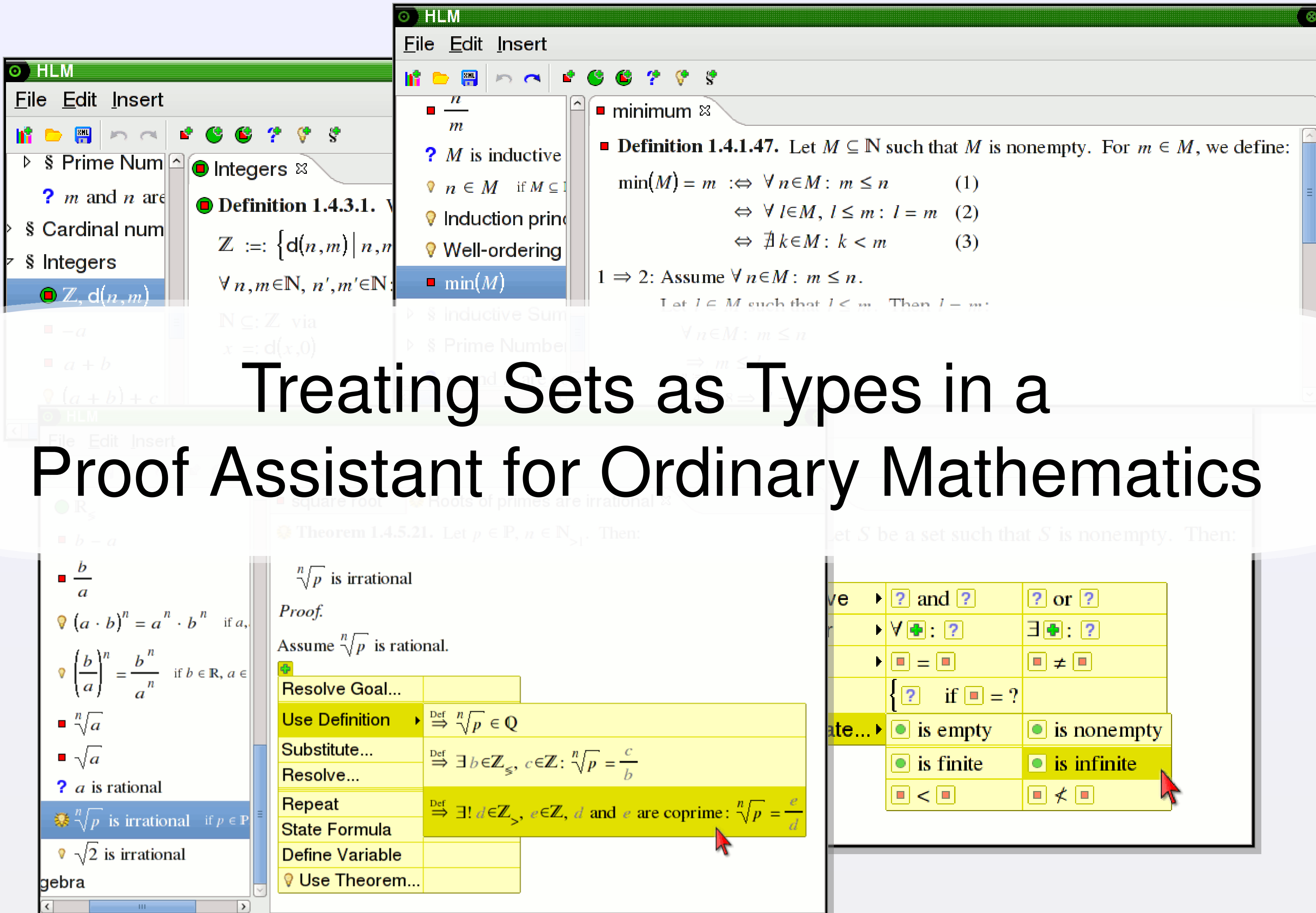


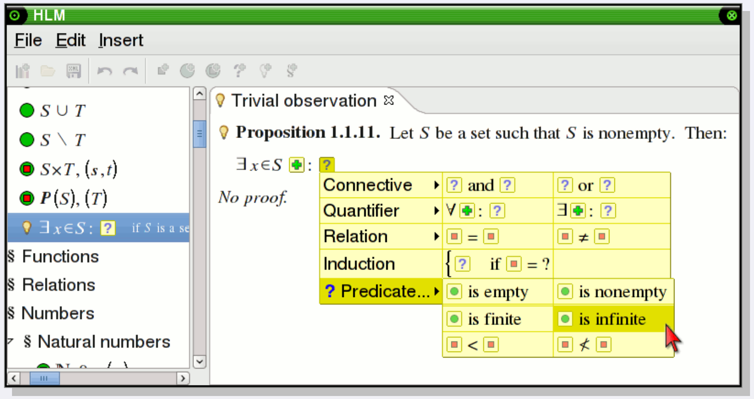
Treating Sets as Types in a Proof Assistant for Ordinary Mathematics

Sebastian Reichelt

Input Methods

Text

GUI



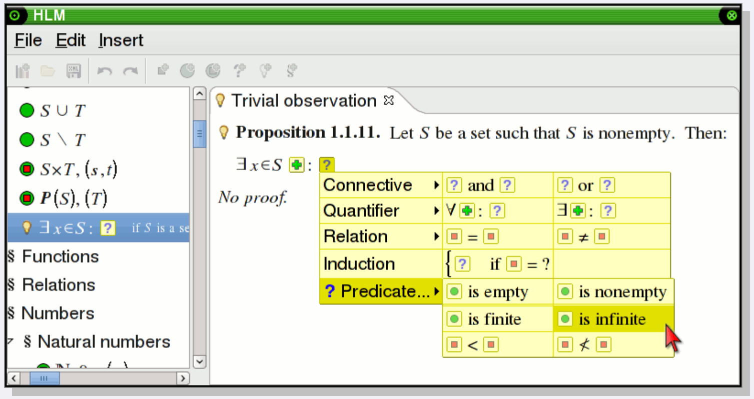
Input Methods

Text

Syntax
Naming
Symbol overloading

GUI

Rendering
Selection



Input Methods

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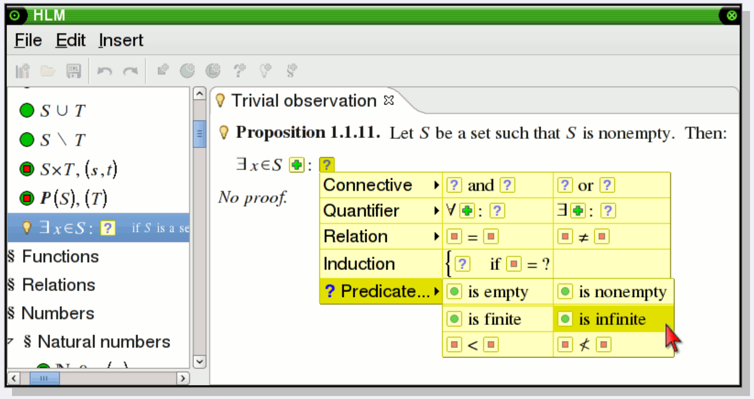
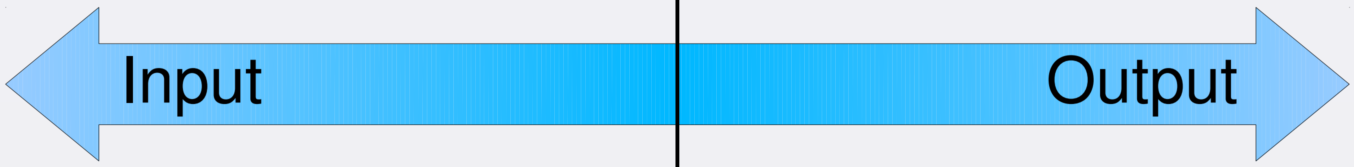
Selection





Input Methods

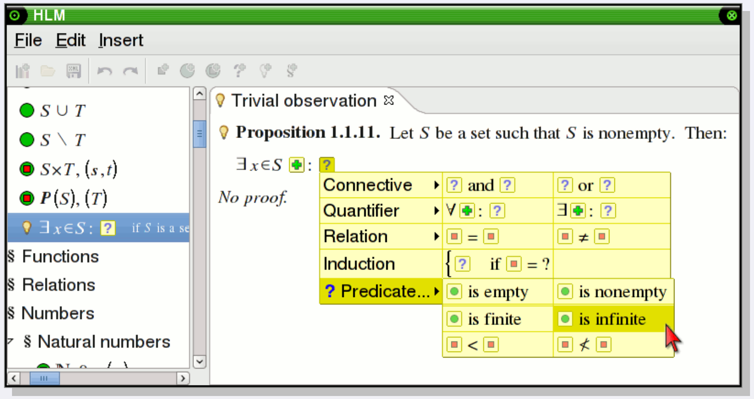
Text

Syntax
Naming
Symbol overloading

Set theory with soft types (→ Mizar)

GUI

Rendering
Selection



Input ↔ Output

Input Methods

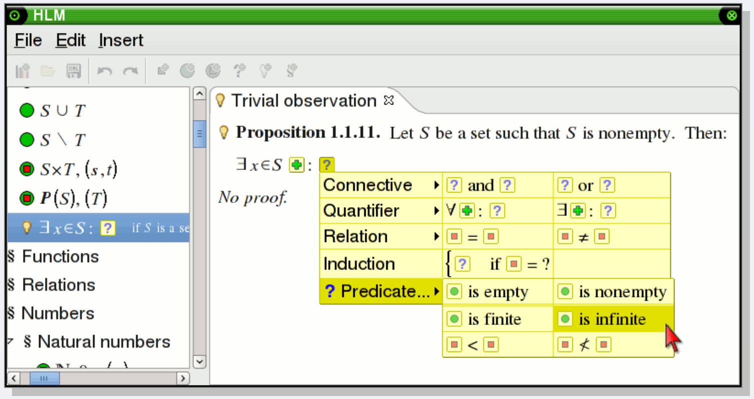
Text

Syntax
Naming
Symbol overloading

Set theory with soft types (→ Mizar)

GUI

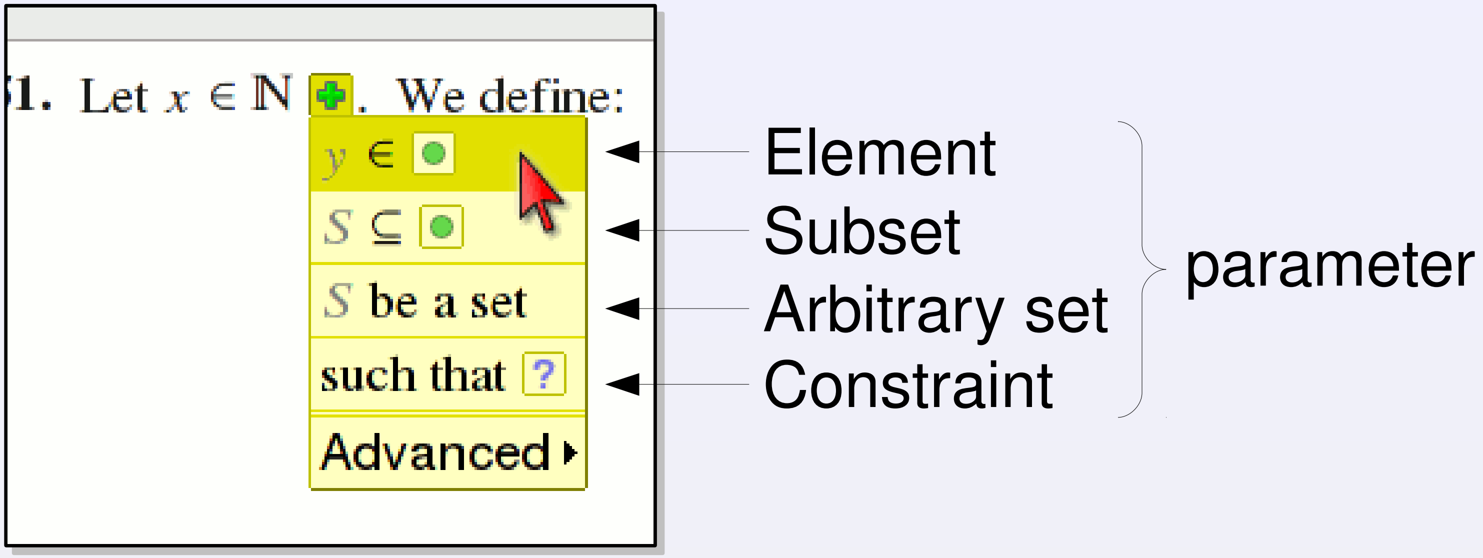
Rendering — *Set theory*
Selection — *Type theory*



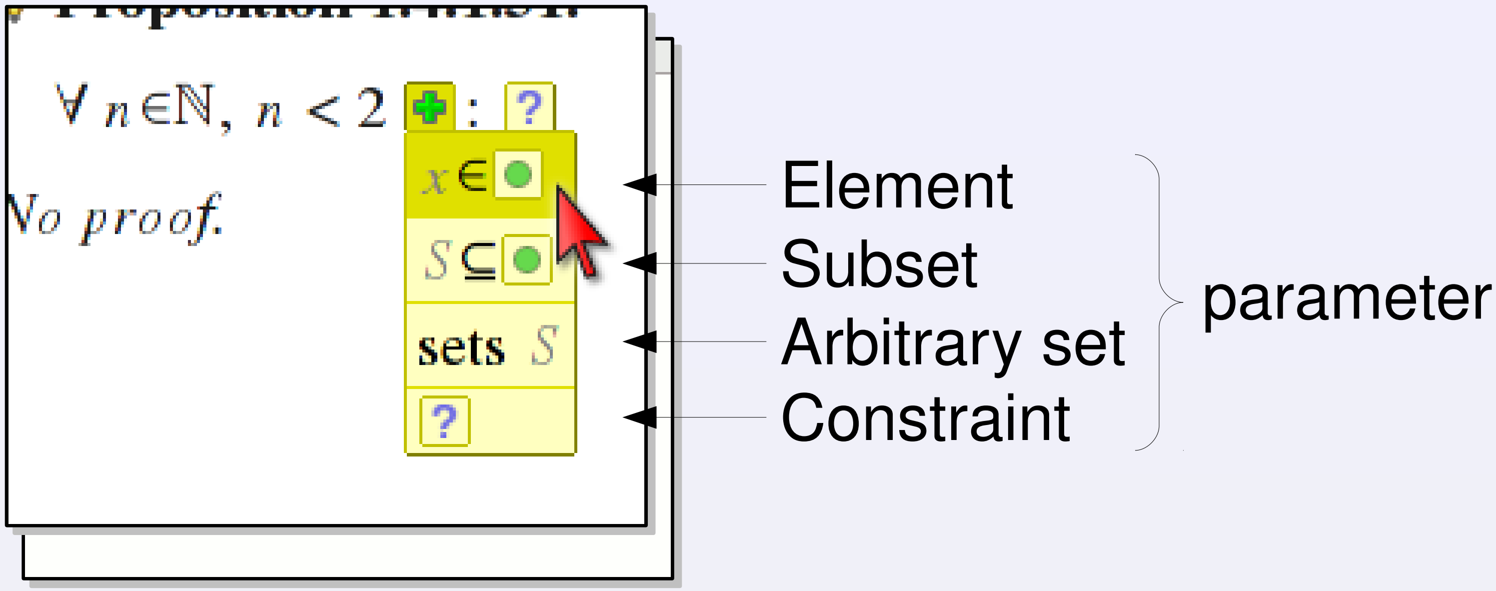
"Theoretically, it seems to be perfectly legitimate to ask whether the union of the cosine function and the number *e* contains a finite geometry" – de Bruijn

Demo

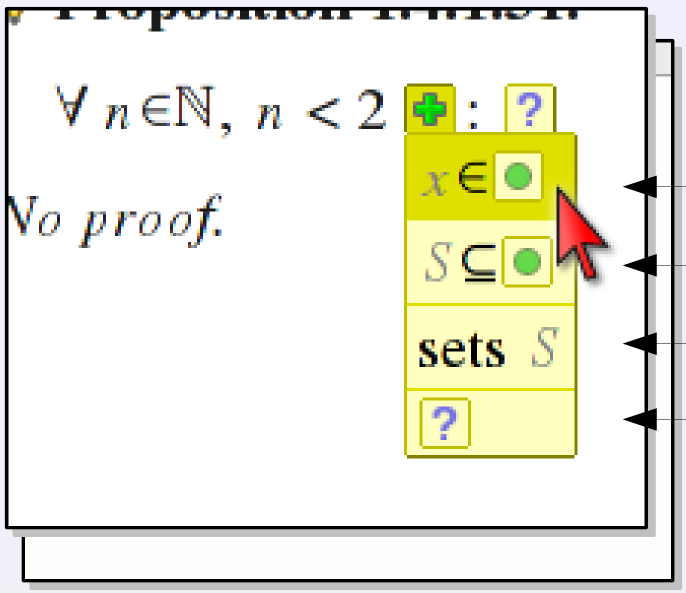
Parameter Lists



Parameter Lists



Parameter Lists

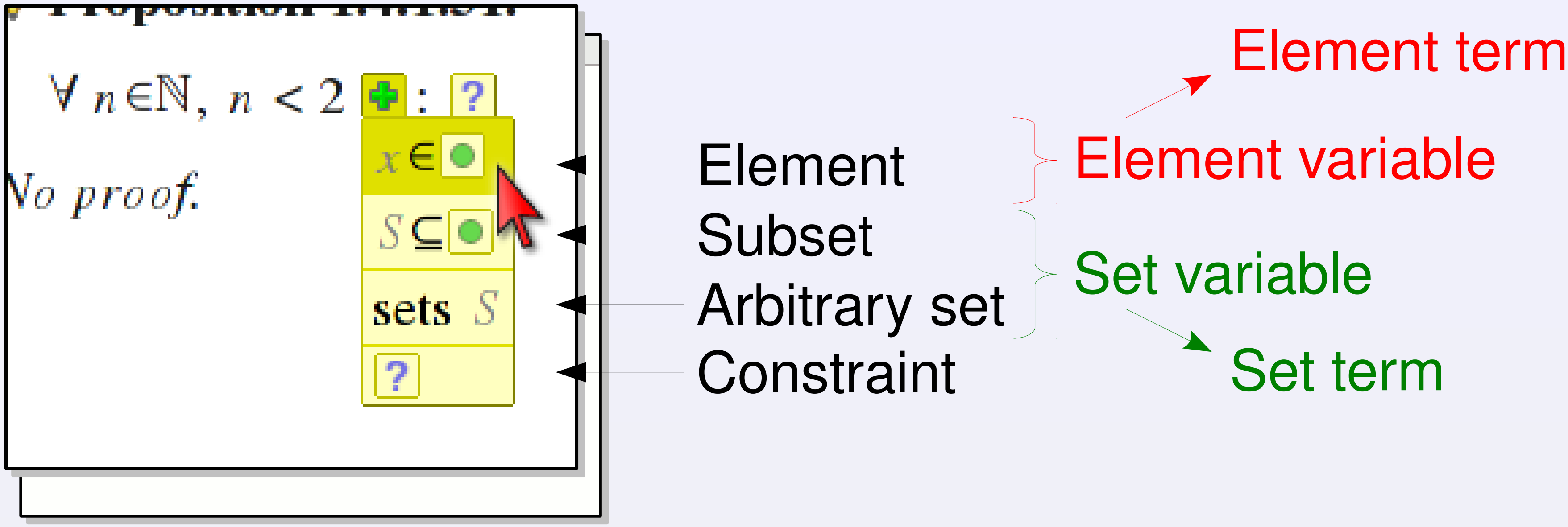


- Element
- Subset
- Arbitrary set
- Constraint

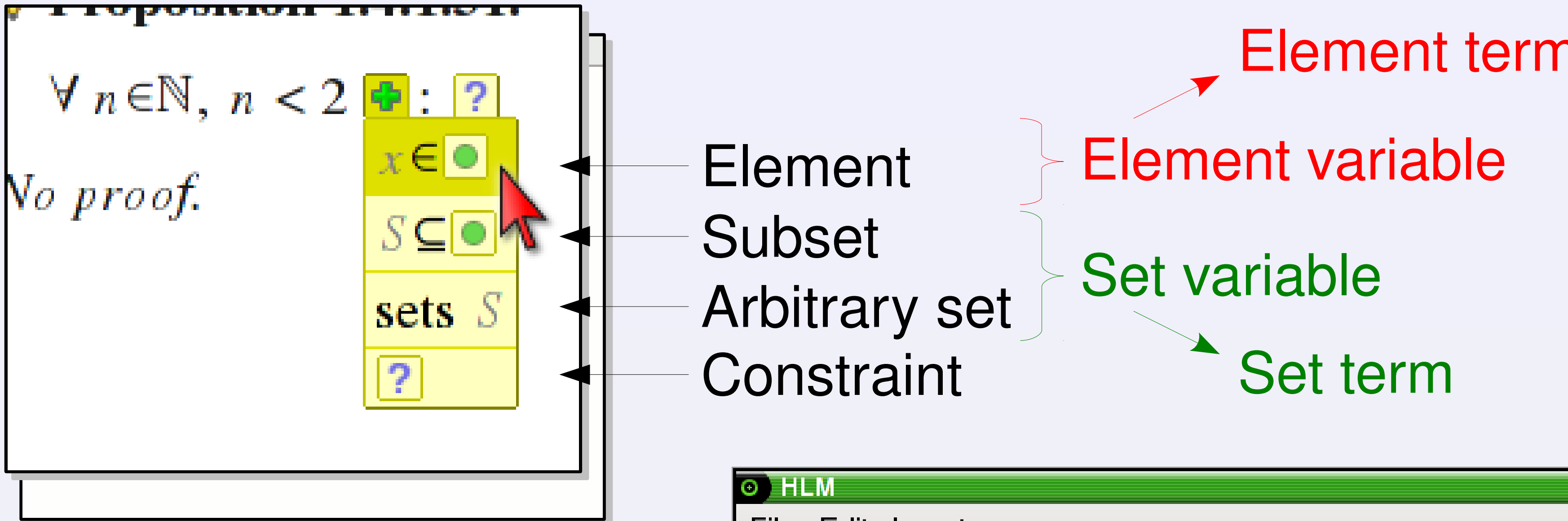
Element variable

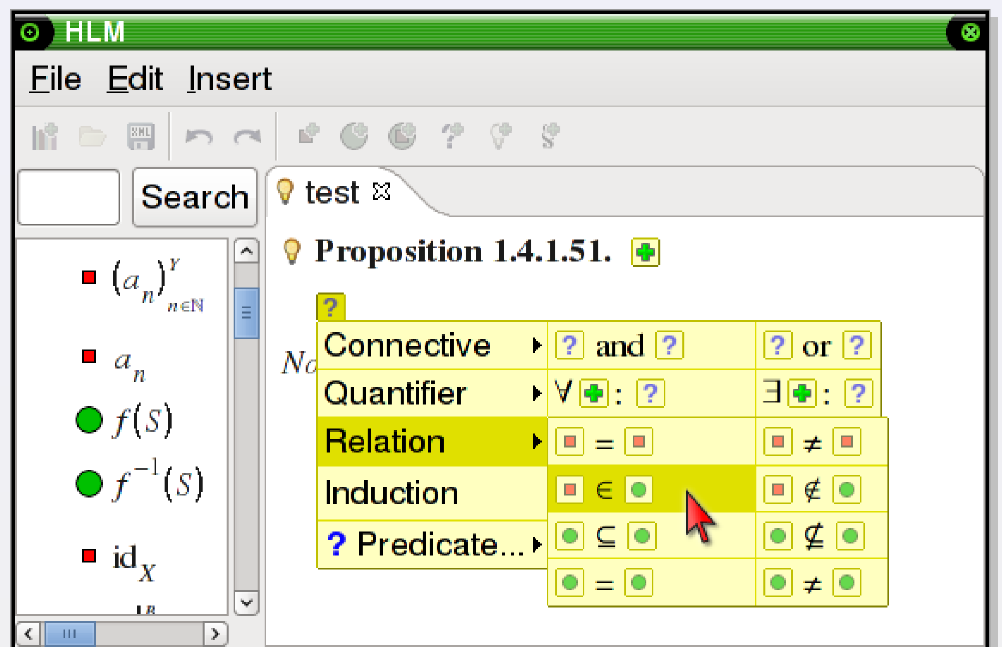
Set variable

Parameter Lists



Parameter Lists





Types

Informal principle:

"x=y" is valid iff x and y are syntactically known to be members of the same set.

Types

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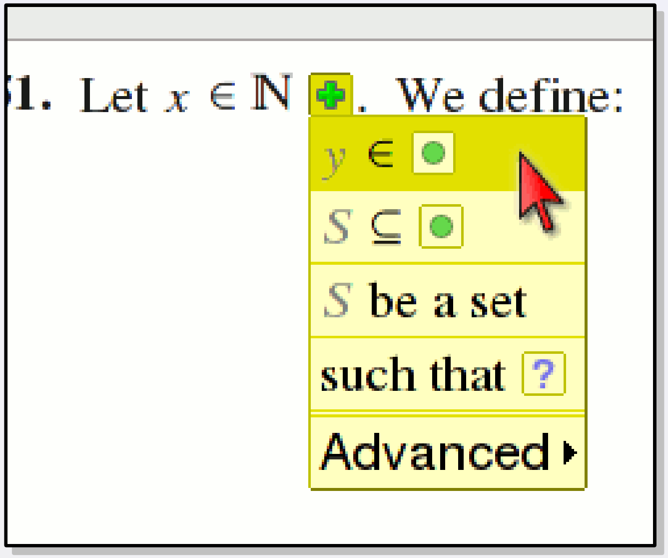
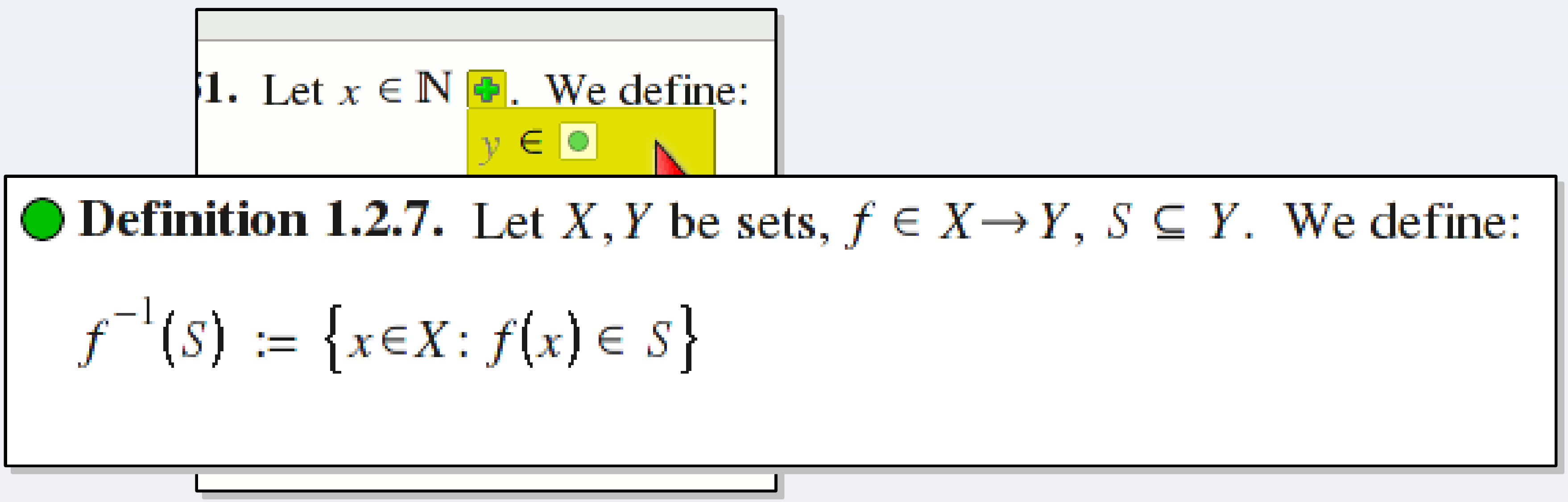


Types

Informal principle:

> "*x*=*y*" is valid iff *x* and *y* are syntactically known to be members of the same set.



Types

Informal principle:

"x=y" is valid iff x and y are syntactically known to be members of the same set.

1. Let $x \in \mathbb{N}$. We define:

$y \in$

Definition 1.2.7. Let X, Y be sets, $f \in X \to Y$, $S \subseteq Y$. We define:

$$f^{-1}(S) := \{x \in X : f(x) \in S\}$$

Y Y

Types

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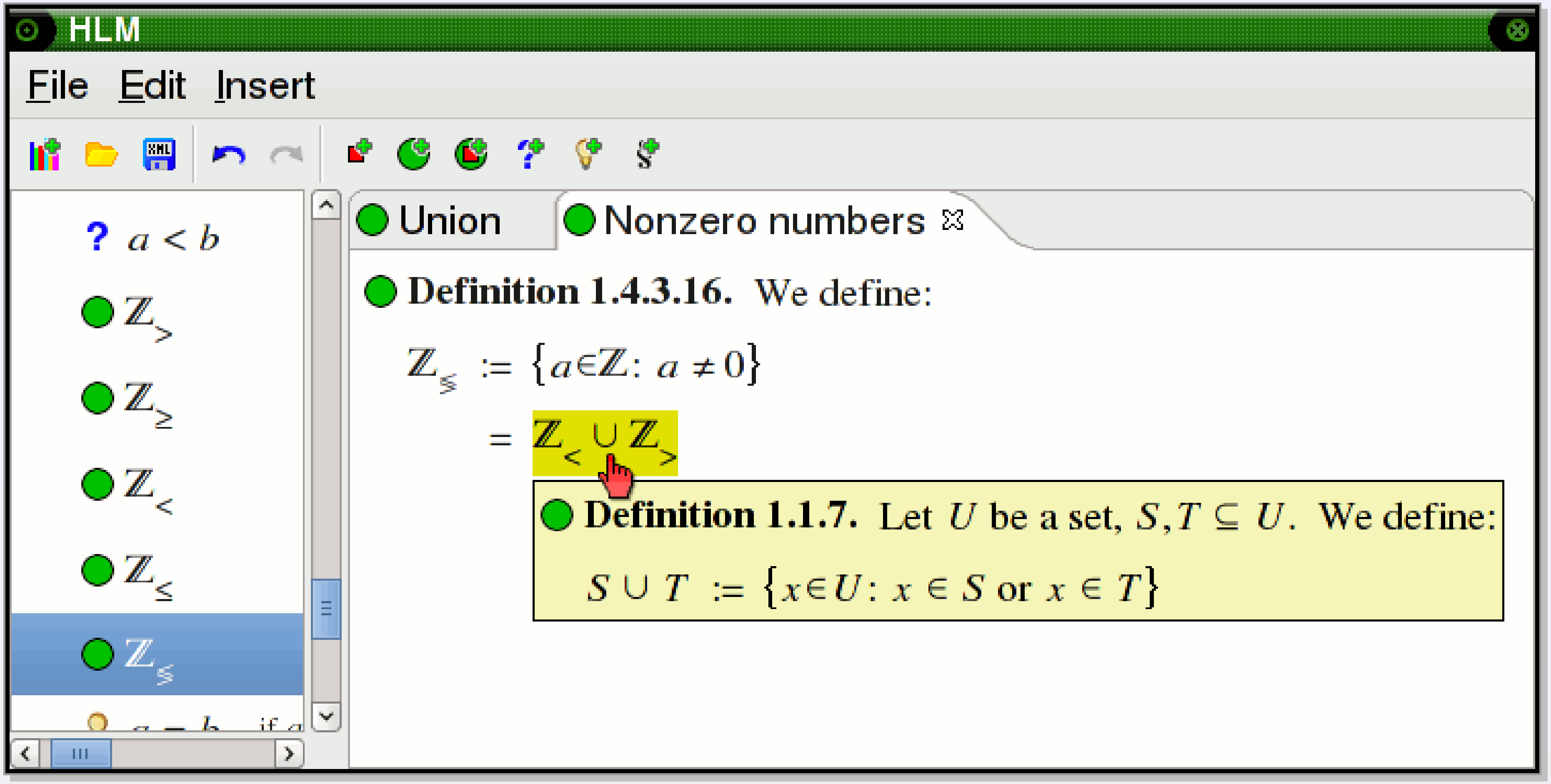
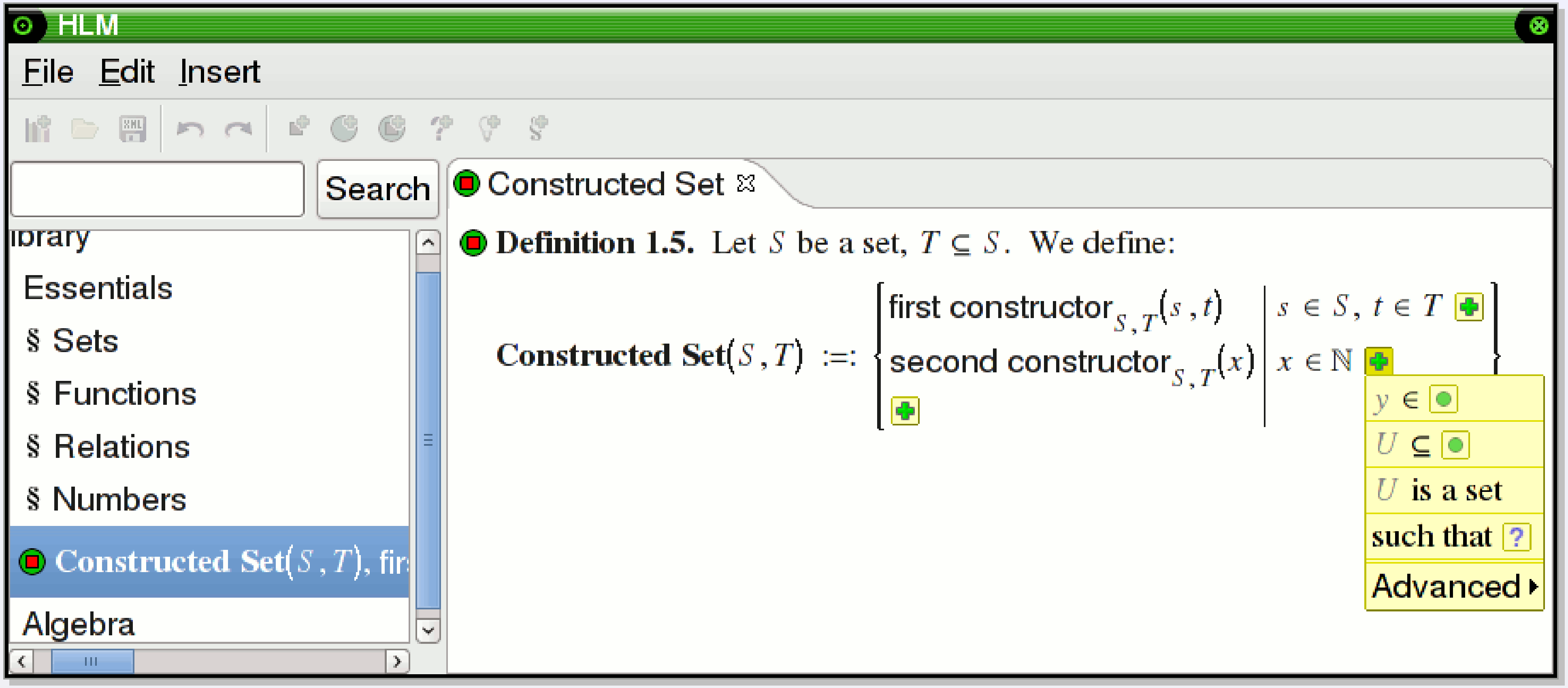
Set Operations



Set Construction



Set Construction

Definition 1.1.9. Let S, T be sets. We define:

$$\textbf{Cartesian product}(S, T) :=: \left\{ \mathsf{pair}_{S,T}(s, t) \;\middle|\; s \in S,\ t \in T \right\}$$

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

$$S \times T := \left\{ (s, t) \,\middle|\, s \in S,\ t \in T \right\}$$

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

S

Definition 1.4.1.1. We define:

$$\mathbb{N} := \left\{ \begin{matrix} 0 \\ \mathsf{s}(n) \end{matrix} \middle| \; n \in \mathbb{N} \right\}$$

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

S

Definition 1.4.1.1. We define:

$\mathbb{N}$

Definition 1.1.10. Let S be a set. We define:

$$\boldsymbol{P}(S) := \left\{ (T) \,\middle|\, T \subseteq S \right\}$$

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

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Definition 1.4.1.1. We define:

$\mathbb{N}$

Definition 1.1.10. Let S be a set. We define:

$$P(S) \;:=:\; \left\{ (T) \,\middle|\, T \subseteq S \right\}$$

Definition 1.1.11. We define:

$$\mathbf{Sets} \;:=:\; \left\{ \mathsf{set}(S) \,\middle|\, S \text{ is a set} \right\}$$

[redacted] $\mathsf{set}(S) = \mathsf{set}(S')$ [redacted]

?

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

Definition 1.4.1.1. We define:

Definition 1.1.10. Let S be a set. We define:

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$\forall$ sets S, S': $\mathrm{set}(S) = \mathrm{set}(S') \;:\Leftrightarrow$?

Set Construction

Definition 1.1.9. Let S, T be sets. We define:

S

Definition 1.4.1.1. We define:

$\mathbb{N}$

Definition 1.1.10. Let S be a set. We define:

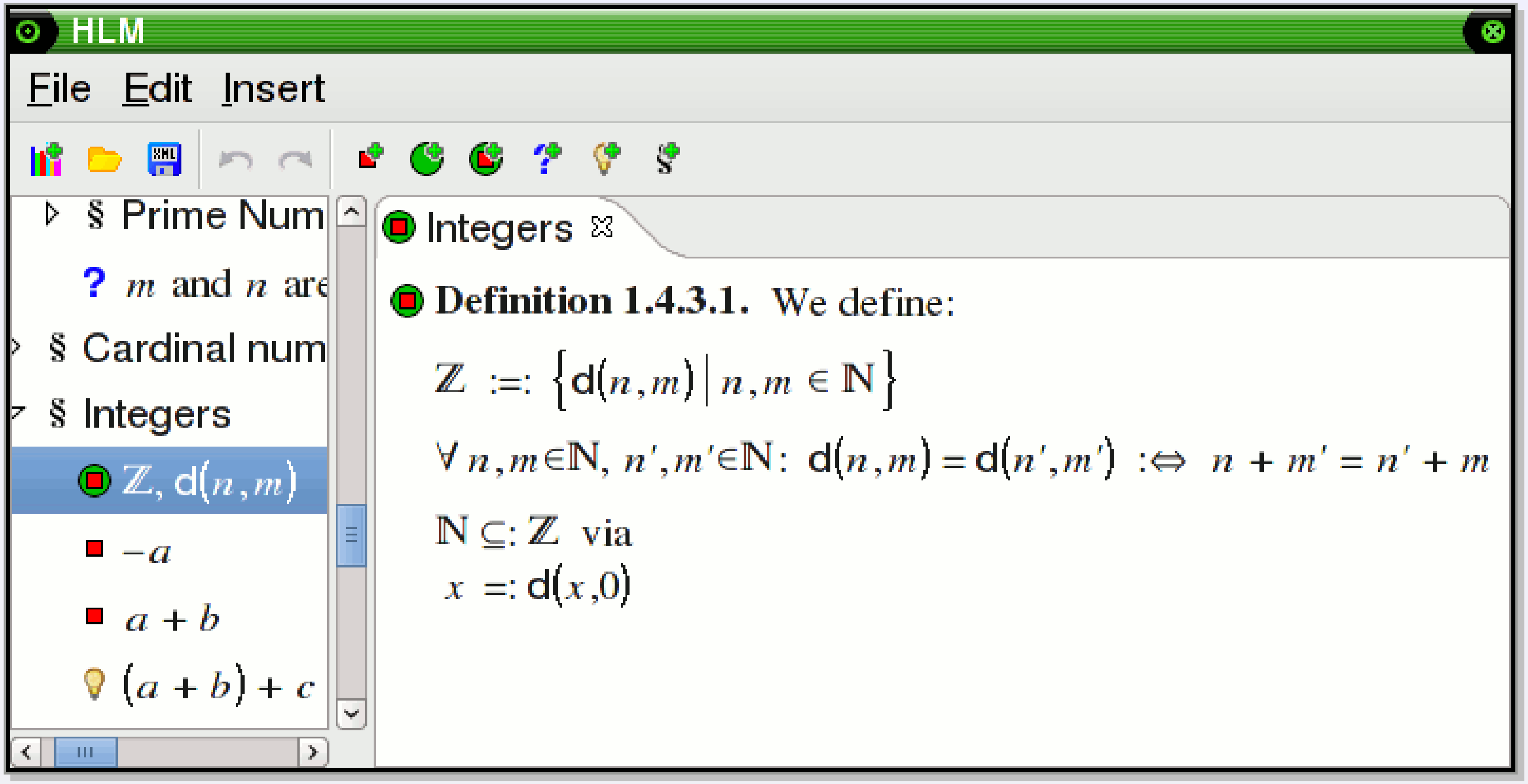
$$\boldsymbol{P}(S) \;:=:\; \left\{ (T) \;\middle|\; T \subseteq S \right\}$$

Definition 1.1.11. We define:

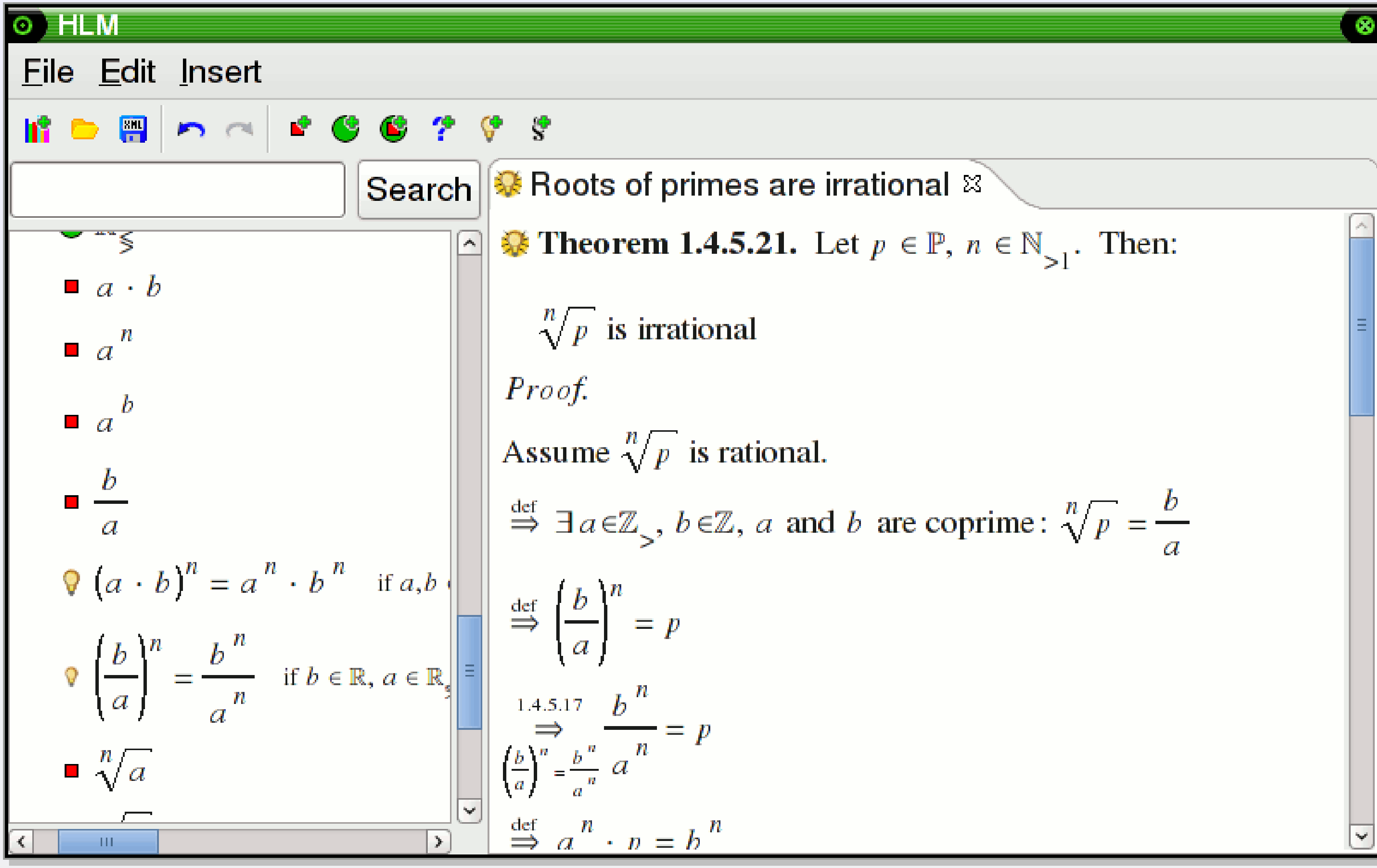
$$\textbf{Sets} \;:=:\; \left\{ \mathsf{set}(S) \;\middle|\; S \text{ is a set} \right\}$$

$$\forall \text{sets } S, S' : \; \mathsf{set}(S) = \mathsf{set}(S') \;:\Leftrightarrow\; \exists f \in S \leftrightarrow S'$$

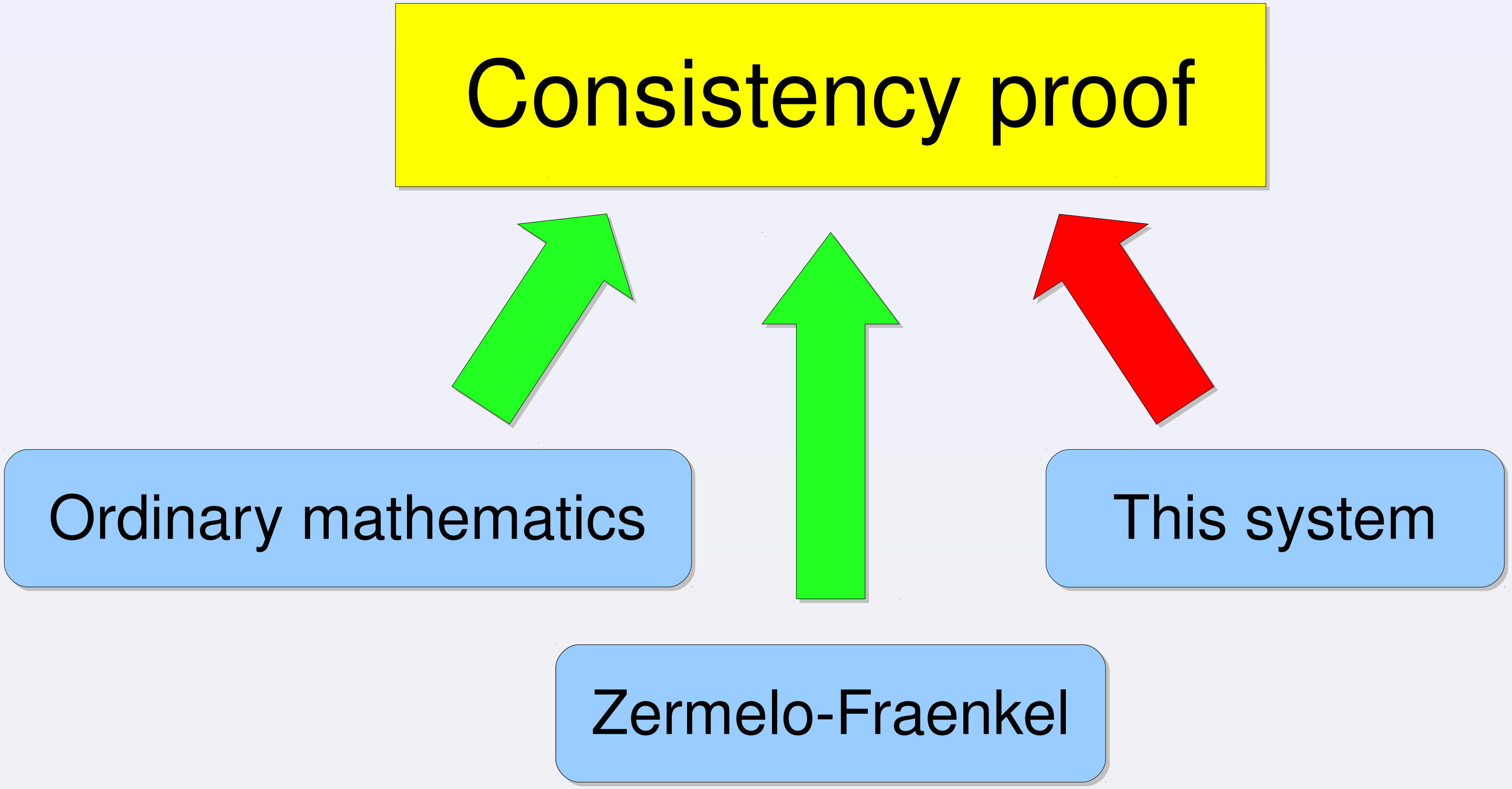
Embedding


HLM
File Edit Insert
§ Prime Num
? m and n are
§ Cardinal num
§ Integers
ℤ, d(n, m)
−a
a + b
(a + b) + c
Integers
Definition 1.4.3.1. We define:
ℤ := {d(n,m) | n,m ∈ ℕ}
∀ n,m∈ℕ, n′,m′∈ℕ: d(n,m) = d(n′,m′) :⇔ n + m′ = n′ + m
ℕ ⊆: ℤ via
x =: d(x,0)

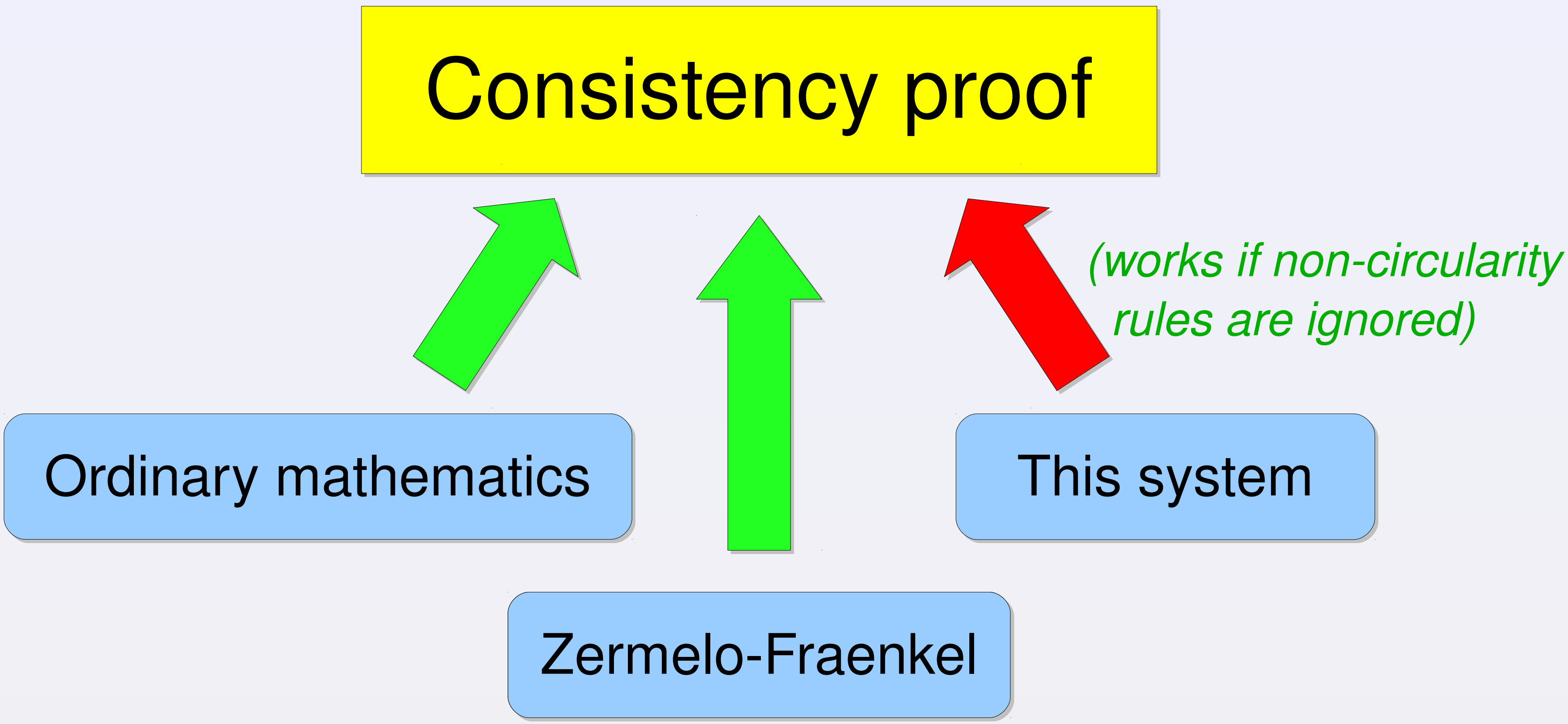
Example


HLM
File
Edit
Insert
Search
Roots of primes are irrational
Theorem 1.4.5.21. Let $p \in \mathbb{P}$, $n \in \mathbb{N}_{>1}$. Then:
$\sqrt[n]{p}$ is irrational
Proof.
Assume $\sqrt[n]{p}$ is rational.
$\overset{def}{\Rightarrow} \exists a \in \mathbb{Z}_{>}, b \in \mathbb{Z}$, a and b are coprime : $\sqrt[n]{p} = \frac{b}{a}$
$\overset{def}{\Rightarrow} \left(\frac{b}{a}\right)^n = p$
$\overset{1.4.5.17}{\Rightarrow} \frac{b^n}{a^n} = p$
$\left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$
$\overset{def}{\Rightarrow} a^n \cdot p = b^n$
$a \cdot b$
a^n
a^b
$\frac{b}{a}$
$(a \cdot b)^n = a^n \cdot b^n$ if a,b
$\left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ if $b \in \mathbb{R}$, $a \in \mathbb{R}$
$\sqrt[n]{a}$

Consistency



Consistency



Conclusions

- Proof assistant modeling mathematical practice
  - Looks like naive set theory
  - Restriction to meaningful inputs (→ types)
- Parameter lists in definitions, theorems, quantifiers, and constructors
- Sets of all structures up to isomorphism
- Embedding
- Consistency

Thank you!

Prototype:
http://hlm.sourceforge.net/