Lambda Calculus with Types

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Lambda Calculus with Types (698 pp)

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Part 1. **Simple Types** $\boldsymbol{\lambda}_{\rightarrow}^{\mathbb{A}}$

Gilles Dowek
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Part 2. **Recursive Types** $\boldsymbol{\lambda}_{=}^{\mathcal{A}}$

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Part 3 **Intersection Types** $\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$

Mariangiola Dezani-Ciancaglini
Fabio Alessi
Furio Honsell
Paula Severi
Pawel Urzyczyn

The fathers

Alonzo Church (1903-1995)
as mathematics student
at Princeton University (1922 or 1924)

Haskell B. Curry (1900-1982)
as BA in mathematics
at Harvard (1920)

Church's contribution: untyped lambda terms (1933)

Lambda terms

$$\boxed{\begin{array}{rcl} \texttt{var} & ::= & \texttt{c} \mid \texttt{var}' \\ \texttt{term} & ::= & \texttt{var} \mid \texttt{term term} \mid \lambda\texttt{var term} \end{array}}$$

Lambda calculus

$$\boxed{(\lambda x.M)N \quad = \quad M[x:=N]}$$

mathematical axiom

$$\boxed{\begin{array}{lcl} & & M = M \\ M = N & \Rightarrow & N = M \\ M = N \;\&\; N = L & \Rightarrow & M = L \\ M = N & \Rightarrow & MP = NP \\ M = N & \Rightarrow & PM = PN \\ M = N & \Rightarrow & \lambda x.M = \lambda x.N \end{array}}$$

logical axiom and rules

We write $\vdash_{\lambda} M = N$ if $M = N$ is provable by these axioms and rules

- Computations termination
- Processes continuation

$\Big\}$ Functional programming (Lisp, Scheme, ML, Clean, Haskell)

Curry's contributions

Curry: Russell paradox as fixed point of the operator Not

Idea: For $a{\in}A$ write Aa

Then as $\{x \mid P[x]\}$ we can take $\lambda x.P[x]$

Indeed, we get the intended interpretation

$a{\in}\{x \mid P[x]\}$ becomes $(\lambda x.P[x])a = P[a]$

Taking $R = \{x \mid x \notin x\} = \lambda x.\neg(xx)$ we get

$$\forall r.[Rr \iff \neg(rr)]$$

hence $RR \iff \neg(RR)$

Note that $RR \equiv (\lambda x.\neg(xx))(\lambda x.\neg(xx)) = \mathsf{Y}(\neg)$ ■

Typing of Whitehead-Russell was transformed into 'functionality'

$$\frac{\Gamma \vdash FABM \quad \Gamma \vdash AN}{\Gamma \vdash B(MN)} \quad \frac{\Gamma, Ax \vdash BM}{\Gamma \vdash FAB(\lambda x.M)}$$

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Typing of Whitehead-Russell was transformed into 'functionality'

$$\frac{\Gamma \vdash M : A{\to}B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : (A{\to}B)}$$

Curry: combinators, correspondence with logic, linguistics

I	$\triangleq$	$\lambda x.x$	:	$A \rightarrow A$
K	$\triangleq$	$\lambda xy.x$	:	$A \rightarrow B \rightarrow A$
S	$\triangleq$	$\lambda xyz.xz(yz)$	:	$(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$

From these all closed lambda terms can be defined applicatively

Also with types

Curry: “Hey, these are tautologies” $\mapsto$ Curry-Howard correspondence

Inspired by Ajdukiewicz (and indirectly by Leśniewski)

Curry gave types to syntactic categories

n noun/subject s sentence

n→n	‘red hat’ (adjective)	(n→s)→(n→s)	adverbs
(n→n)→n	‘redness’	(n→s)→s	quantifiers
n→(n→n)	‘(John and Henry) are brothers’		
n→s	‘Mary sleeps’		
n→n→s	‘Mary kisses John’		
s→s	‘not(Mary kisses John)’		
More complex cases			
(n→n)→(n→n)→(n→n)		‘slightly large’	
((n→n)→(n→n))→(n→n)→(n→n)		‘slightly too large’	

Principia Mathematica (Whitehead-Russell 1910)

$*50{\cdot}16.$ $\vdash . I``\alpha = \alpha$

Dem.

$$\begin{array}{ll} & \vdash . *37{\cdot}1 . \supset \vdash : x \in I``\alpha . \equiv . (\exists y) . y \in \alpha . xIy . \\ [*50{\cdot}1] & \qquad\qquad \equiv . (\exists y) . y \in \alpha . x = y . \\ [*13{\cdot}195] & \qquad\qquad \equiv . x \in \alpha : \supset \vdash . \mathrm{Prop} \end{array}$$

$*50{\cdot}59.$ $\vdash . (I \upharpoonright \alpha)``\beta = \alpha \cap \beta$

Dem.

$$\begin{array}{ll} & \vdash . *37{\cdot}412 . \supset \vdash . (I \upharpoonright \alpha)``\beta = I``(\alpha \cap \beta) \\ [*50{\cdot}16] & \qquad\qquad = \alpha \cap \beta . \supset \vdash . \mathrm{Prop} \end{array}$$

Substitution is needed

PM does not provide it: λ-calculus does

The cycle

Untyped lambda terms (6)		$\boldsymbol{\lambda}$	$\langle \Lambda, \cdot \rangle$
Simple types (22)	Free type algebras	$\boldsymbol{\lambda}^{\mathbb{A}}_{\rightarrow}$	$\langle \mathbb{A}, \rightarrow \rangle$
Recursive types (2)	Type algebras	$\boldsymbol{\lambda}^{\mathcal{A}}_{=}$	$\langle \mathcal{A}, \rightarrow, = \rangle$
Subtyping (1)	Type structures	$\boldsymbol{\lambda}^{\mathcal{S}}_{\leq}$	$\langle \mathcal{S}, \rightarrow, \leq \rangle$
Intersection types (4)	Intersection type structures	$\boldsymbol{\lambda}^{\mathcal{S}}_{\cap}$	$\langle \mathcal{S}, \rightarrow, \leq, \cap, \top \rangle$

All untyped lambda terms appear again

$\boldsymbol{\lambda}^{\mathbb{A}}_{\to}$ An increasing chain of systems $\boldsymbol{\lambda}^{\mathbb{A}}_{\to} \subseteq \boldsymbol{\lambda}^{\mathcal{A}}_{=} \subseteq \boldsymbol{\lambda}^{\mathcal{S}}_{\leq} \subseteq \boldsymbol{\lambda}^{\mathcal{S}}_{\cap}$

- $\boldsymbol{\lambda}^{\mathcal{S}}_{\cap}$
  - $\boldsymbol{\lambda}^{\mathcal{S}}_{\leq}$
    - $\boldsymbol{\lambda}^{\mathcal{A}}_{=}$
      - $\boldsymbol{\lambda}^{\mathbb{A}}_{\to}$

$$\Gamma, x : A \vdash x : A$$

$$\frac{\Gamma \vdash M : (A{\to}B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A{\to}B)}$$

$$\frac{\Gamma \vdash M : A \quad A = B}{\Gamma \vdash M : B}$$

$$\frac{\Gamma \vdash M : A \quad A \leq B}{\Gamma \vdash M : B}$$

$$\frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$$

$$\Gamma \vdash M : \top$$

$\boldsymbol{\lambda}^{\mathbb{A}}_{\rightarrow}$ An increasing chain of systems $\boldsymbol{\lambda}^{\mathbb{A}}_{\rightarrow} \subseteq \boldsymbol{\lambda}^{\mathcal{A}}_{=} \subseteq \boldsymbol{\lambda}^{\mathcal{S}}_{\leq} \subseteq \boldsymbol{\lambda}^{\mathcal{S}}_{\cap}$

$\boldsymbol{\lambda}^{\mathbb{A}}_{\rightarrow}$

$$\Gamma, x : A \vdash x : A$$

$$\frac{\Gamma \vdash M : (A {\rightarrow} B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A {\rightarrow} B)}$$

$\boldsymbol{\lambda}^{\mathcal{A}}_{=}$

$$\frac{\Gamma \vdash M : A \quad A = B}{\Gamma \vdash M : B}$$

$\boldsymbol{\lambda}^{\mathcal{S}}_{\leq}$

$$\frac{\Gamma \vdash M : A \quad A \leq B}{\Gamma \vdash M : B}$$

$\boldsymbol{\lambda}^{\mathcal{S}}_{\cap}$

$$\frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$$

$$\Gamma \vdash M : \top$$

$\boldsymbol{\lambda}_{\to}^{\mathbb{A}}$ Examples

$\boldsymbol{\lambda}_{\to}^{\mathbb{A}}$	$\lambda xy.xyy : (A{\to}A{\to}B){\to}A{\to}B$	
$\boldsymbol{\lambda}_{=}^{\mathcal{A}}$	$\lambda x.xx : A$ $(\lambda x.xx)(\lambda x.xx) : B$	if $A = A{\to}B$ in $\mathcal{A}$
$\boldsymbol{\lambda}_{\leq}^{\mathcal{S}}$	$\lambda x.xx : A{\to}B$ $\lambda x.xx : (A{\to}B){\to}B$	only; if $A \leq A{\to}B$ in $\mathcal{S}$ if $A{\to}B \leq A$
$\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$	$\lambda x.xx : A \cap (A{\to}B){\to}B$ $\mathsf{KI}\Omega : A{\to}A$	where $\Omega \triangleq (\lambda x.xx)(\lambda x.xx)$ as $\Omega : \top$

Simple types from ground type 0

$$\boxed{\mathbb{T} \quad = \quad 0 \mid \mathbb{T} \rightarrow \mathbb{T}}$$

$\Lambda(A)$: λ-terms of type A. Write $\Lambda_{\rightarrow} = \bigcup_{A \in \mathbb{T}} \Lambda(A)$

$$\begin{array}{rcl} & & x^A \in \Lambda(A) \\ M \in \Lambda(A \rightarrow B), N \in \Lambda(A) & \Rightarrow & (MN) \in \Lambda(B) \\ M \in \Lambda(B) & \Rightarrow & (\lambda x^A.M) \in \Lambda(A \rightarrow B) \end{array}$$

Church's version of $\boldsymbol{\lambda}_{\rightarrow}^{\mathbb{A}}$

Default equality $=_{\boldsymbol{\beta\eta}}$ preserves types

$$\begin{array}{rcll} (\lambda x^A.M)N & = & M[x^A := N] & \boldsymbol{\beta}\text{-conversion} \\ \lambda x^A.Mx^A & = & M & \boldsymbol{\eta}\text{-conversion} \end{array}$$

Prop. (i) For $M \in \Lambda_{\rightarrow}^{\emptyset}(A)$ one has

$$\vdash |M| : A$$

(ii) For $M \in \Lambda^{\emptyset}$ in $\boldsymbol{\beta}$-nf such that $\vdash M : A$

there is a unique $M^A \in \Lambda(A)$ such that $|M^A| \equiv M$

(iii) For open M not in $\boldsymbol{\beta}$-nf (ii) fails: $\mathsf{KI}y : A \rightarrow A$

(iv) Even for closed M not in $\boldsymbol{\beta}$-nf (ii) fails: $(\lambda x.x\mathsf{I})(\lambda y.\mathsf{I}) : A \rightarrow A$

The counter-examples in (iii), (iv) are due to the presence or creation of a K-redex

Prop. For normal M one can identify $\vdash M : A$ and $M^A \in \Lambda(A)$

preserving reduction

Prop. $\vdash M : A$	$\Rightarrow$	M has a $\boldsymbol{\beta\eta}$-nf	(Normalization Theorem)
Prop. $\vdash M : A$ & $M \twoheadrightarrow_{\boldsymbol{\beta\eta}} M'$	$\Rightarrow$	$\vdash M' : A$	(Subject Reduction Theorem)

Def. Let $\mathcal{M} = \{\mathcal{M}(A)\}_{A\in\mathbb{T}}$ be a family of non-empty sets

(i) $\mathcal{M}$ is called a *type structure* for $\lambda_{\to}^{o}$ if

$$\mathcal{M}(A{\to}B) \subseteq \mathcal{M}(B)^{\mathcal{M}(A)}$$

Here Y^X denotes the collection of set-theoretic functions

$$\{f \mid f : X \to Y\}$$

(ii) Let $\mathcal{M}$ be provided with *application operators*

$$\begin{array}{rcl} (\mathcal{M}, \cdot) & = & (\{\mathcal{M}(A)\}_{A\in\mathbb{T}}, \{\cdot_{A,B}\}_{A,B\in\mathbb{T}}) \\ \cdot_{A,B} & : & \mathcal{M}(A{\to}B) \times \mathcal{M}(A) \to \mathcal{M}(B). \end{array}$$

A *typed applicative structure* is such an $(\mathcal{M}, \cdot)$ satisfying *extensionality*:

$$\forall f, g{\in}\mathcal{M}(A{\to}B)\, [[\forall a{\in}\mathcal{M}(A)\ f \cdot_{A,B} a = g \cdot_{A,B} a] \ \Rightarrow\ f = g].$$

Prop. The notions ‘type structure’ and ‘typed applicative structure’ are equivalent

Def. Given a set X. The *full type structure* over X

$$\mathcal{M}_X = \{X(A)\}_{A\in\mathbb{T}}$$

where $X(A)$ is defined inductively as follows

$$\begin{aligned} X(0) &\triangleq X; \\ X(A{\to}B) &\triangleq X(B)^{X(A)}, \text{ the set of functions from } X(A) \text{ into } X(B) \end{aligned}$$

Def. $\mathcal{M}_n \triangleq \mathcal{M}_{\{1,\cdots,n\}}$

$\ldots$	
$\mathcal{M}(A{\to}B)$	F
$\ldots$	
$\mathcal{M}(A)$	M
$\ldots$	
$\mathcal{M}(B)$	FM
$\ldots$	
$\mathcal{M}(2)$	
$\mathcal{M}(1) = X^X$	
$\mathcal{M}(0) = X$	

Partial view of $\mathcal{M} = \mathcal{M}_X$
$F, M\in\Lambda^{\mathrm{ch}}_{\to}$

finite at each level A if X is finite

Def. Given a set X. The *full type structure* over X

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$\ldots$	
$\mathcal{M}(A{\to}B)$	$[\![F]\!]$
$\ldots$	
$\mathcal{M}(A)$	$[\![M]\!]$
$\ldots$	
$\mathcal{M}(B)$	$[\![FM]\!]$
$\ldots$	
$\mathcal{M}(2)$	
$\mathcal{M}(1) = X^X$	
$\mathcal{M}(0) = X$	

Partial view of $\mathcal{M} = \mathcal{M}_X$
$F, M \in \Lambda_{\to}^{\mathrm{ch}}$

finite at each level A if X is finite

Let ρ be a valuation in $\mathcal{M}_X$: we require $\rho(x^A) \in \mathcal{M}(A)$

For $M \in \Lambda_{\rightarrow}(A)$ we define $[\![M]\!]_\rho \in \mathcal{M}(A)$

$$\begin{array}{rcl}
[\![x^A]\!]_\rho & \triangleq & \rho(x^A) \\
[\![MN]\!]_\rho & \triangleq & [\![M]\!]_\rho [\![N]\!]_\rho \\
[\![\lambda x^A.M]\!]_\rho & \triangleq & \lambda\!\!\lambda d \in X(A).[\![M]\!]_{\rho(x^A:=d)}
\end{array}$$

where $\rho(x^A := d) = \rho'$ with

$$\begin{array}{rcll}
\rho'(x^A) & \triangleq & d & \\
\rho'(y^B) & \triangleq & \rho(y^B) & \quad \text{if } y^B \not\equiv x^A
\end{array}$$

Define

$$\begin{array}{rcll}
\mathcal{M}_X \models M = N & \overset{\triangle}{\Longleftrightarrow} & \forall \rho \; [\![M]\!]_\rho = [\![N]\!]_\rho & \\
& \overset{\triangle}{\Longleftrightarrow} & [\![M]\!] = [\![N]\!] & \quad \text{if } M, N \in \Lambda_{\rightarrow}^{\emptyset}
\end{array}$$

We know that $\mathbb{N} = \langle \mathbb{N}, +, \times, 0, 1 \rangle$ can be 'λ-defined'

Prop. The rational numbers

$$\mathbb{Q} = \langle \mathbb{Q}, +, \times, -, :, 0, 1 \rangle$$

cannot be λ-defined: there is no type Q and terms

$$\begin{array}{lcl} \vdash M_{+} & : & Q \to Q \to Q \\ \vdash M_{\times} & : & Q \to Q \to Q \\ \vdash M_{-} & : & Q \to Q \to Q \\ \vdash M_{:} & : & Q \to Q \to Q \\ \vdash M_{0} & : & Q \\ \vdash M_{1} & : & Q \end{array}$$

such that the usual laws hold

Proof. The homomorphic image of a field K is K itself;

but $\mathcal{M}_X(Q)$ is finite for X finite ■

Def. $\mathsf{Th}(\mathcal{M}) = \{M = N \mid M, N \in \Lambda^{\emptyset}_{\to} \;\&\; \mathcal{M} \models M = N\}$

$\mathsf{Th}(\mathcal{M}_1)$ is inconsistent: all terms of the same type are equated

$\mathcal{M}_2 \models \mathbf{c}_1 = \mathbf{c}_3 : 1{\to}0{\to}0$

$\mathcal{M}_2 \not\models \mathbf{c}_1 = \mathbf{c}_2 : 1{\to}0{\to}0$

Exercises

$\mathcal{M}_2 \models \mathbf{c}_2 = \mathbf{c}_4 = \mathbf{c}_6 = \cdots$

$\mathcal{M}_2 \models \mathbf{c}_1 = \mathbf{c}_3 = \mathbf{c}_5 = \cdots$

$\mathcal{M}_2 \not\models \mathbf{c}_0 = \mathbf{c}_1$

$\mathcal{M}_2 \not\models \mathbf{c}_0 = \mathbf{c}_2$

$\mathcal{M}_5 \models \mathbf{c}_4 = \mathbf{c}_{64}$ $\quad$ $(64 = \mathrm{lcm}\{1, 2, 3, 4, 5\})$

$\mathcal{M}_6 \not\models \mathbf{c}_4 = \mathbf{c}_{64}$ $\quad$ even if $(64 = \mathrm{lcm}\{1, 2, 3, 4, 5, 6\})$

$\mathcal{M}_6 \models \mathbf{c}_5 = \mathbf{c}_{65}$

$\boldsymbol{\lambda}^{\mathbb{A}}_{\rightarrow}$ Partial semantics (Friedman [1975])

Let $\mathcal{M}$ be a typed applicative structure

A *partial valuation* in $\mathcal{M}$ is a family $\rho = \{\rho_A\}_{A\in\mathbb{T}}$ of partial maps

$$\rho_A : \mathtt{Var}(A) \hookrightarrow \mathcal{M}(A)$$

The *partial semantics* $[\![\ ]\!]^{\mathcal{M}}_{\rho} : \Lambda_{\rightarrow}(A) \hookrightarrow \mathcal{M}(A)$ under ρ is

$$\begin{aligned}
[\![x^A]\!]^{\mathcal{M}}_{\rho} &\triangleq \rho_A(x) \\
[\![PQ]\!]^{\mathcal{M}}_{\rho} &\triangleq [\![P]\!]^{\mathcal{M}}_{\rho}[\![Q]\!]^{\mathcal{M}}_{\rho} \\
[\![\lambda x^A.P]\!]^{\mathcal{M}}_{\rho} &\triangleq \lambda\!\!\lambda d{\in}M(A).[\![P]\!]^{\mathcal{M}}_{\rho[x:=d]}
\end{aligned}$$

Often we write $[\![M]\!]_{\rho}$ for $[\![M]\!]^{\mathcal{M}}_{\rho}$

The expression $[\![M]\!]_{\rho}$ may not always be defined, even if ρ is total

The problem arises with $[\![\lambda x.P]\!]_{\rho}$ when

$$\lambda\!\!\lambda d{\in}M(A).[\![P]\!]^{M}_{\rho[x:=d]} \in \mathcal{M}(B)^{\mathcal{M}(A)} - \mathcal{M}(A{\rightarrow}B)$$

If $[\![\lambda x.P]\!]_{\rho}$ exists it is uniquely defined

A *typed λ-model* is a type structure $\mathcal{M}$ such that

$$[\![M]\!]_\rho \text{ is defined}$$

for all $A \in \mathbb{T}$, $M \in \Lambda(A)$, and ρ such that $\mathrm{FV}(M) \subseteq \mathrm{dom}(\rho)$

Examples of typed λ-models

- $\mathcal{M}_X$: full type structures
- $\mathcal{M}_{\boldsymbol{\beta\eta}}$: open typed terms modulo $\boldsymbol{\beta\eta}$-equality
- $\mathcal{M}[\mathcal{C}]$: closed typed term models modulo extensionality

where $\mathcal{C}$ is a set of typed constants such that $\mathcal{M}[\mathcal{C}](0) \neq \emptyset$

In $\mathcal{M}[\mathcal{C}]$ one can <u>define</u>

$$[\![M]\!]_\rho = [M[\vec{x} := \rho(\vec{x})]]$$

and show it works

There are only five $\mathsf{Th}(\mathcal{M}[\mathcal{C}])$ coming from

$$\mathcal{C}_1 = \{\boldsymbol{c}^0, \boldsymbol{d}^0\}$$

$$\mathcal{C}_2 = \{\boldsymbol{c}^0, \boldsymbol{f}^1\}$$

$$\mathcal{C}_3 = \{\boldsymbol{c}^0, \boldsymbol{f}^1, \boldsymbol{g}^1\}$$

$$\mathcal{C}_4 = \{\boldsymbol{c}^0, \boldsymbol{\Phi}^{3\to 0\to 0}\}$$

$$\mathcal{C}_5 = \{\boldsymbol{c}^0, \boldsymbol{b}^{0\to 0\to 0}\}$$

There is a sixth, the inconsistent theory, coming from $\mathcal{C}_0 = \{\boldsymbol{c}^0\}$

One has

$$\mathsf{Th}(\mathcal{M}[\mathcal{C}_5]) \subseteq \mathsf{Th}(\mathcal{M}[\mathcal{C}_4]) \subseteq \mathsf{Th}(\mathcal{M}[\mathcal{C}_3]) \subseteq \mathsf{Th}(\mathcal{M}[\mathcal{C}_2]) \subseteq \mathsf{Th}(\mathcal{M}[\mathcal{C}_1]) \subseteq \mathsf{Th}(\mathcal{M}[\mathcal{C}_0])$$

$\mathsf{Th}(\mathcal{M}[\mathcal{C}_5]) = \{M = N \mid M =_{\boldsymbol{\beta\eta}} N\}$ minimal theory

$\mathsf{Th}(\mathcal{M}[\mathcal{C}_1])$ is the unique maximally consistent theory

consisting of all consistent equations together

$\mathcal{M}[\mathcal{C}_1]$ is the minimal model, with decidable equality (Loader)

Let $\mathcal{C}$ be a set of typed constants

Examples $\mathcal{C}_0 = \{\boldsymbol{c}^0\}$, $\mathcal{C}_1 = \{\boldsymbol{c}^0, \boldsymbol{d}^0\}$

Define $\Lambda_{\rightarrow}^{\emptyset}[\mathcal{C}]$ as the set of closed typed λ-terms built up from $\mathcal{C}$

Now $\Lambda_{\rightarrow}^{\emptyset}[\mathcal{C}]$ modulo 'extensionality' will be considered as a term model

For $M, N \in \mathcal{M}[\mathcal{C}](A)$ define

$$
\begin{array}{rcll}
M \approx_{\mathcal{C}}^{\mathrm{ext}} N & \stackrel{\Delta}{\Longleftrightarrow} & M =_{\boldsymbol{\beta\eta}} N & \text{if } A = 0 \\
& \stackrel{\Delta}{\Longleftrightarrow} & \forall P \in \mathcal{M}[\mathcal{C}](B).[MP \approx_{\mathcal{C}}^{\mathrm{ext}} NP] & \text{if } A = B \rightarrow C
\end{array}
$$

Then $M \approx_{\mathcal{C}}^{\mathrm{ext}} N \iff \forall \vec{P} \in \mathcal{M}[\mathcal{C}].[M\vec{P} =_{\boldsymbol{\beta\eta}} N\vec{P}]$

Thm. $\mathcal{M}[\mathcal{C}] = \Lambda_{\rightarrow}^{\emptyset}[\mathcal{C}]/\approx_{\mathcal{C}}^{\mathrm{ext}}$ is an extensional λ-model

not trivial at all, we need

$$\mathcal{M}[\mathcal{C}] \models M = N \;\Rightarrow\; \underline{\mathcal{M}[\mathcal{C}] \models FM = FN} \;\&\; \mathcal{M}[\mathcal{C}] \models \lambda x.M = \lambda x.N$$

Exercises 1. $\mathcal{M}(\boldsymbol{c}^0) \cong \mathcal{M}_1$, where $\mathcal{M}[\boldsymbol{c}^0] = \mathcal{M}[\{\boldsymbol{c}^0\}]$

2. $\mathcal{M}[\boldsymbol{c}^0, \boldsymbol{d}^0] \models \mathbf{c}_1 = \mathbf{c}_2$

3. $\mathcal{M}[\boldsymbol{c}^0, \boldsymbol{d}^0] \not\cong \mathcal{M}_2$

For $M, N \in \Lambda_{\to}^{\emptyset}[\mathcal{C}](A)$ define

$$M \approx_{\mathcal{C}}^{\mathrm{obs}} N \stackrel{\Delta}{\iff} \forall F : (A \to 0). FM =_{\boldsymbol{\beta\eta}} FN$$

Remark

$$M \approx_{\mathcal{C}}^{\mathrm{obs}} N \quad \Rightarrow \quad FM \approx_{\mathcal{C}}^{\mathrm{obs}} FN$$

$$M \approx_{\mathcal{C}}^{\mathrm{obs}} N \quad \Rightarrow \quad M \approx_{\mathcal{C}}^{\mathrm{ext}} N$$

$$M \approx_{\mathcal{C}}^{\mathrm{ext}} N \quad \iff \quad \forall Z. MZ \approx_{\mathcal{C}}^{\mathrm{ext}} NZ$$

Thm. $\forall M, N.[M \approx_{\mathcal{C}}^{\mathrm{obs}} N \iff M \approx_{\mathcal{C}}^{\mathrm{ext}} N]$ non-trivial

Cor. $\mathcal{M}[\mathcal{C}]$ is an (extensional) λ-model

A *relation* on $\mathcal{M}[\mathcal{C}]$ is a family $R = \{R_A\}_{A\in\mathbb{T}}$ with $R_A \subseteq \mathcal{M}[\mathcal{C}](A)^n$

A relation is *logical* if for all $A, B\in\mathbb{T}$ and all $\vec{M}\in\mathcal{M}[\mathcal{C}](A\to B)^n$

$$\begin{aligned} R_{A\to B}(M_1,\cdots,M_n) \quad \iff \quad & \forall N_1\in\mathcal{M}[\mathcal{C}](A)\cdots N_n\in\mathcal{M}[\mathcal{C}](A) \\ & [R_A(N_1,\cdots,N_n) \;\Rightarrow\; R_B(M_1N_1,\cdots,M_nN_n)] \end{aligned}$$

Thus a logical relation is fully determined by R_0

Prop. Suppose $\approx_{\mathcal{C}}^{\text{ext}}$ is logical on $\mathcal{M}[\mathcal{C}]$. Then for all $M, N\in\mathcal{M}[\mathcal{C}]$

$$M\approx_{\mathcal{C}}^{\text{ext}}N \iff M\approx_{\mathcal{C}}^{\text{obs}}N$$

Proof. Only ($\Rightarrow$) is interesting

Assume $M\approx_{\mathcal{C}}^{\text{ext}}N$ and $F\in\mathcal{M}[\mathcal{C}](A\to 0)$ towards $FM =_{\boldsymbol{\beta\eta}} FN$

Trivially $F\approx_{\mathcal{C}}^{\text{ext}}F$

$$\begin{aligned} &\Rightarrow \quad FM\approx_{\mathcal{C}}^{\text{ext}}FN, && \text{as } \approx_{\mathcal{C}}^{\text{ext}} \text{ is logical} \\ &\Rightarrow \quad FM =_{\boldsymbol{\beta\eta}} FN, && \text{as the type is } 0 \;\blacksquare \end{aligned}$$

It remains to show that the $\approx_{\mathcal{C}_i}^{\mathtt{ext}}$ are logical

Def. BE is the logical relation on $\mathcal{M}[\mathcal{C}]$ determined by

$$\mathsf{BE}_0(M, N) \overset{\Delta}{\Longleftrightarrow} M =_{\boldsymbol{\beta\eta}} N$$

Lemma 1. Suppose $\mathsf{BE}(\boldsymbol{c}, \boldsymbol{c})$ for $\boldsymbol{c} \in \mathcal{C}$. Then

$$\forall M \in \Lambda[\mathcal{C}].\mathsf{BE}(M, M)$$

Proof. By the usual arguments for logical relations. ■

Lemma 2. Suppose BE(c, c) for all $c \in \mathcal{C}$. Then $\approx_{\mathcal{C}}^{\text{ext}}$ is BE and hence logical

Proof. By Lemma 1 one has for all $M \in \mathcal{M}[\mathcal{C}]$

$$\mathsf{BE}(M, M) \tag{0}$$

It follows that BE is an equivalence relation on $\mathcal{M}[\mathcal{C}]$. We claim that for all $F, G \in \mathcal{M}[\mathcal{C}](A)$

$$\mathsf{BE}_A(F, G) \iff F \approx_{\mathcal{C}}^{\text{ext}} G,$$

By induction on the structure of A. Case $A = 0$. By definition. Case $A = B \to C$, then

$$\begin{array}{llcll}
(\Rightarrow) & \mathsf{BE}_{B\to C}(F, G) & \Rightarrow & \mathsf{BE}_C(FP, GP), & \text{for all } P \in \mathcal{M}[\mathcal{C}](B), \\
 & & & \multicolumn{2}{l}{\text{since } P \approx_{\mathcal{C}}^{\text{ext}} P \text{ and hence by the IH } \mathsf{BE}_B(P, P)} \\
 & & \Rightarrow & FP \approx_{\mathcal{C}}^{\text{ext}} GP, & \text{for all } P \in \mathcal{M}[\mathcal{C}] \text{ by the IH} \\
 & & \Rightarrow & F \approx_{\mathcal{C}}^{\text{ext}} G, & \text{by definition.} \\
(\Leftarrow) & F \approx_{\mathcal{C}}^{\text{ext}} G & \Rightarrow & FP \approx_{\mathcal{C}}^{\text{ext}} GP, & \text{for all } P \in \mathcal{M}[\mathcal{C}], \\
 & & \Rightarrow & \mathsf{BE}_C(FP, GP) & \qquad\qquad (1)
\end{array}$$

by the induction hypothesis. In order to prove BE$_{B\to C}(F, G)$, assume BE$_B(P, Q)$ towards BE$_C(FP, GQ)$. Well, since also BE$_{B\to C}(G, G)$, by (0), we have

$$\mathsf{BE}_C(GP, GQ). \tag{2}$$

It follows from (1) and (2) and the transitivity of BE (which on this type is the same as $\approx_{\mathcal{C}}^{\text{ext}}$ by the IH) that BE$_C(FP, GQ)$ indeed.

By the claim $\approx_{\mathcal{C}}^{\text{ext}}$ is BE and therefore $\approx_{\mathcal{C}}^{\text{ext}}$ is logical. ■

Lemma 3. BE(M, M) holds for $M \in \mathcal{M}[\mathcal{C}]$ of types $0, 1, 0\to 0\to 0$. Proof. Easy. ■

Lemma 4. Let $\boldsymbol{c} = \boldsymbol{c}^3 \in \mathcal{M}[\mathcal{C}]$. Suppose

$$\forall F, G \in \mathcal{M}[\mathcal{C}](2)[F \approx^{\text{ext}}_{\mathcal{C}} G \;\Rightarrow\; F =_{\beta\eta} G]$$

Then $\text{BE}_{A\to 0}(\boldsymbol{c}, \boldsymbol{c})$

Proof. Let $\boldsymbol{c}$ be given. Then for $F, G \in \mathcal{M}[\mathcal{C}](2)$, $P \in \mathcal{M}[\mathcal{C}](1)$ one has

$$\begin{array}{rcll}
\text{BE}(F, G) & \Rightarrow & FP =_{\beta\eta} GP & \text{by Lemma 3} \\
 & \Rightarrow & F \approx^{\text{ext}}_{\mathcal{C}} G & \\
 & \Rightarrow & F =_{\beta\eta} G & \text{by assumption} \\
 & \Rightarrow & \boldsymbol{c}F =_{\beta\eta} \boldsymbol{c}G &
\end{array}$$

Therefore we have by definition BE$(\boldsymbol{c}, \boldsymbol{c})$ ■

Last mortgage

For every $F, G \in \mathcal{M}[\mathcal{C}](2)$ one has

$$F \approx^{\texttt{ext}}_{\mathcal{C}_4} G \;\Rightarrow\; F =_{\beta\eta} G.$$

We must show

$$[\forall h \in \mathcal{M}[\mathcal{C}](1). Fh =_{\beta\eta} Gh] \;\Rightarrow\; F =_{\beta\eta} G. \tag{1}$$

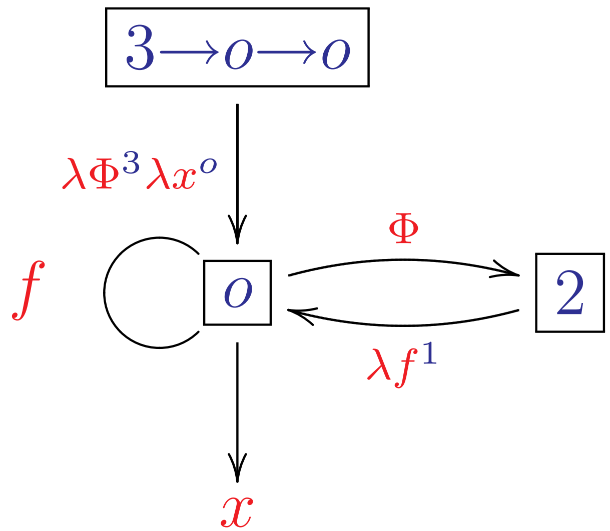
$\lambda_{\rightarrow}^{\mathbb{A}}$ Analysis of terms of given type

$3{\rightarrow}0{\rightarrow}0$ $\quad \lambda\Phi x.x, \lambda\Phi x.\Phi(\lambda f.x), \lambda\Phi x.\Phi(\lambda f.fx), \lambda\Phi x.\Phi(\lambda f.f(\Phi(\lambda g.g(fx)))), \cdots$

$\lambda\Phi x.\Phi(\lambda f_1.w_{\{f_1\}}x), \lambda\Phi x.\Phi(\lambda f_1.w_{\{f_1\}}\Phi(\lambda f_2.w_{\{f_1,f_2\}}x)), \cdots;$

$\boldsymbol{\lambda\Phi x.\Phi(\lambda f_1.w_{\{f_1\}}\Phi(\lambda f_2.w_{\{f_1,f_2\}}\cdots\Phi(\lambda f_n.w_{\{f_1,\cdots,f_n\}}x)\cdot\cdot))} \triangleq$

$`\langle w_{\{f_1\}}, w_{\{f_1,f_2\}}, \cdots, w_{\{f_1,\cdots,f_n\}}\rangle'$



Let $h_m \triangleq \lambda x.\Phi(\lambda f.f^m x) = `\langle f^m\rangle' : \mathcal{M}[\mathcal{C}](1)$

Claim

$$\forall F,G\in\mathcal{M}[\mathcal{C}](2)\exists m\in\mathbb{N}.[Fh_m = Gh_m \;\Rightarrow\; F =_{\beta\eta} G]$$

That the $\mathsf{Th}(\mathcal{M}[\mathcal{C}_i])$ form a chain follows from *type reducibility*

Def. $A \leq_{\beta\eta} B \stackrel{\Delta}{\iff} \exists F{:}A{\rightarrow}B$

$$\forall M_1 M_2{:}A.[M_1 =_{\beta\eta} M_2 \iff FM_1 =_{\beta\eta} FM_2]$$

("there is a λ-definable injection from A to B")

Thm. [Hierarchy Theorem (Statman [1980])] Inhabited members of $\mathbb{T}$ can be partitioned in decidable classes $\mathbb{T}_0, \mathbb{T}_1, \cdots, \mathbb{T}_5$ such that

$$\begin{array}{llll}
0 & <_{\beta\eta} & 0{\rightarrow}0 & \in\mathbb{T}_0 \\
 & <_{\beta\eta} & 0^2{\rightarrow}0 & \\
 & <_{\beta\eta} & \cdots & \\
 & <_{\beta\eta} & 0^k{\rightarrow}0 & \in\mathbb{T}_1 \\
 & <_{\beta\eta} & \cdots & \\
 & <_{\beta\eta} & 1{\rightarrow}0{\rightarrow}0 & \in\mathbb{T}_2 \\
 & <_{\beta\eta} & 1{\rightarrow}1{\rightarrow}0{\rightarrow}0 & \in\mathbb{T}_3 \\
 & <_{\beta\eta} & 3{\rightarrow}0{\rightarrow}0 & \in\mathbb{T}_4 \\
 & <_{\beta\eta} & (0^2{\rightarrow}0){\rightarrow}0{\rightarrow}0 & \in\mathbb{T}_5
\end{array}$$

and all $A, B{\in}\mathbb{T}_i$ are $\boldsymbol{\beta\eta}$-equivalent

$A \leq_{\beta\eta} B \;\&\; B \leq_{\beta\eta} A$

All not-inhabited types are equivalent to 0

Thm. Each $\mathcal{M}[\mathcal{C}]$ is a term model provided $\mathcal{M}[\mathcal{C}](0)$ is inhabited

Thm. There are only five (six) resulting theories

Open problems

- Can this be proved more directly?
- Are $\mathsf{Th}(\mathcal{M}[\mathcal{C}_2])$, $\mathsf{Th}(\mathcal{M}[\mathcal{C}_3])$, $\mathsf{Th}(\mathcal{M}[\mathcal{C}_4])$ decidable?

If we want $A = A \to B$ we simply work in $\mathbb{T}$ modulo $A = A \to B$

This leads to recursive types via *simultaneous recursion*

Alternatively we can write $A \triangleq \mu\alpha.\alpha \to B$ and postulate $A = A \to B$

Consider

$$\boxed{\mathbb{T}_{\mu}^{\mathbb{A}} \quad ::= \quad \mathbb{A} \mid \mathbb{T}_{\mu}^{\mathbb{A}} \to \mathbb{T}_{\mu}^{\mathbb{A}} \mid \mu\mathbb{A}\,\mathbb{T}_{\mu}^{\mathbb{A}}}$$

where we work modulo μ-reduction

$$\mu\alpha.A \Rightarrow_{\mu} A[\alpha := \mu\alpha.A]$$

We must be careful not to create confusion of variables

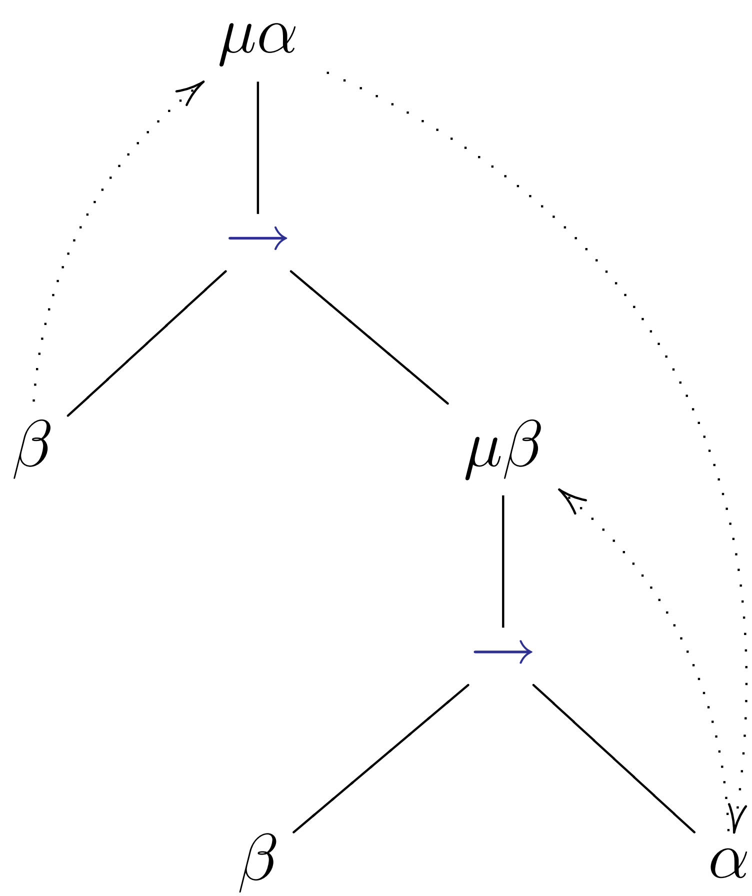
The μ-type $\mu\alpha.(\beta \to \mu\beta.(\alpha \to \beta))$ is not safe:

'naively contracting' $\mu\alpha$ leads to a clash

$\boldsymbol{\lambda}^{\mathcal{A}}_{=}$ Avoiding renaming (alpha-reduction)

Thm. [V. van Oostrom] By contrast to λ-terms for every $A \in \mathbb{T}^{\mathbb{A}}_{\mu}$ one can make a renaming A' such that never a variable clash occurs with the μ-reducts of A'

Proof-sketch. Avoid configurations like



This is hereditary ■

$\boldsymbol{\lambda}_{\leq}^{\mathcal{S}}$ Principal type structure

Prop. (A. Polonsky)

For every $M \in \Lambda^{\emptyset}$ there exists a 'principal pair' $(\mathcal{S}_M, a_M)$ such that

$$\vdash_{\mathcal{S}_M} M : a$$

$$\vdash_{\mathcal{S}} M : a \quad \Longleftrightarrow \quad \exists h : \mathcal{S}_M \to \mathcal{S}.h(a_M) \leq a$$

for all type structures $\mathcal{S} = \langle \mathcal{S}, \to, \leq \rangle$ and $a \in \mathcal{S}$

Def. Given $M, N \in \Lambda^{\emptyset}$ define

$$M \precsim N \overset{\Delta}{\Longleftrightarrow} \text{there exists a morphism } h : \mathcal{S}_M \to \mathcal{S}_N$$

Open problems

- Is $\precsim$ on $\Lambda^{\emptyset}$ decidable?
- Study the unsolvables under $\precsim$

$\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$ Subject Reduction

In $\boldsymbol{\lambda}_{\rightarrow}^{o}$ one has

$$\left.\begin{array}{l}\Gamma \vdash M : A \\ M \twoheadrightarrow_{\beta} N\end{array}\right\} \quad \Rightarrow \quad \Gamma \vdash N : A$$

This also holds for $\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$, for many intersection type structures $\mathcal{S}$

The converse, *subject expansion*, does not hold for $\boldsymbol{\lambda}_{\rightarrow}^{o}$

$\lambda_\cap^{\mathcal{S}}$ Subject expansion

Suppose

$$\vdash P[x := Q] : A$$

where $P \equiv \cdots x \cdots x \cdots x \cdots$

so $\cdots Q \cdots Q \cdots Q \cdots : A$

Each of these occurrences of Q may need another type B_1, B_2, B_3

But then we can give $\lambda x.P$ the type $B_1 \cap B_2 \cap B_3 \to A$

Hence the β-expansion $(\lambda x.P)Q$ also the type A

If the number of occurrences of x in P is 0,

then we may give to $\lambda x.P$ the type $\mathtt{U} \to A$

which is consistent as the *empty* intersection

again

$$\vdash (\lambda x.P)Q : A$$

$\boldsymbol{\lambda}_\cap^{\mathcal{S}}$ A model for $\boldsymbol{\lambda\beta}$ (Barendregt, Coppo, Dezani)

Therefore $\left.\begin{array}{l}\Gamma \vdash M : A \\ M =_{\boldsymbol{\beta}} N\end{array}\right\} \Rightarrow \Gamma \vdash N : A$

so (for closed M)

$$X_M = \{A \mid\ \vdash M : A\}$$

which looks like a λ-model. Indeed, such a set X is a *filter* of types

$$\left| \begin{array}{l} X \neq \emptyset \\ A, B \in X \ \Rightarrow\ (A \cap B) \in X \\ B \geq A \in X \ \Rightarrow\ B \in X \end{array}\right.$$

For filters X, Y one can define application

$$XY = \{B \mid\ \exists A \in Y\ (A \to B) \in X\}$$

is well defined and one has (for many intersection type structures)

$$X_M X_N = X_{MN}$$

Given an intersection type structure $\mathcal{S}$, then $\mathcal{F}^{\mathcal{S}} = \{X \subseteq \mathcal{S} \mid X \text{ is a filter}\}$

is the filter structure over $\mathcal{S}$. If $\mathcal{S}$ is natural it is a λ-model.

Open problems

Investigate equivalence of categories involved

Recent result relating $\boldsymbol{\lambda}_{\rightarrow}^{\mathbb{A}}$ and $\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$

Sylvain Salvati: Elements of $\mathcal{M}_n$ can be described as intersection types

relation between results of Loader and Urzyczyn concerning respectively

- undecidability of λ-definability in $\mathcal{M}_n$
- undecidability of inhabitation in $\boldsymbol{\lambda}_{\cap}^{\mathcal{S}}$

$$
\begin{array}{rcl}
d{\in}\mathcal{M}_X(A) & \leadsto & \xi_d^A{\in}\mathbb{T}_{\cap}^X \\
\xi_d^0 & = & d \\
\xi_d^{A\to B} & = & \displaystyle\bigcap_{e\in X(B)} \xi_e^B{\to}\xi_{de}^A
\end{array}
$$

$$
[\![M]\!] = d \iff \vdash M : \xi_d^A, \qquad \text{for } M{\in}\Lambda(A)
$$

Marketing Strategy (sine qua non)

Summary in ≤ 20 words of 698 pages

This handbook with exercises reveals in formalisms

hitherto mainly used for designing and verifying software and hardware

unexpected mathematical beauty